JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 111, A06306, doi:10.1029/2005JA011546, 2006

Incoherent scatter spectrum theory for modes propagating perpendicular to the geomagnetic field

Erhan Kudeki[1] and Marco Milla[1]

Received 23 November 2005; revised 28 February 2006; accepted 1 March 2006; published 20 June 2006.

[1] The collisional incoherent scatter spectral theory of Woodman is extended to the $T_e/T_i > 1$ case by using its equivalent electron- and ion-Gordeyev integrals in existing collisionless spectral models. A collisional spectral model for $T_e/T_i > 1$ is needed in the interpretation of F region incoherent scatter returns obtained at small magnetic aspect angles.

Citation: Kudeki, E., and M. Milla (2006), Incoherent scatter spectrum theory for modes propagating perpendicular to the geomagnetic field, *J. Geophys. Res.*, *111*, A06306, doi:10.1029/2005JA011546.

1. Introduction

[2] The practitioner of incoherent scatter radar technique is facing a minor dilemma today in the aftermath of papers by *Sulzer and González* [1999] and *Woodman* [2004] on the importance of Coulomb collisions at small magnetic aspect angles α.

[3] The dilemma is as follows: incoherent scatter (IS) radar cross sections are known to be sensitive to electron-ion temperature ratio T_e/T_i at small aspect angles [e.g., *Farley*, 1966], but the only available analytic theory of IS spectrum that accounts for Coulomb collisions, namely, the expressions of *Woodman* [2004] taken from *Woodman* [1967], has been derived for the $T_e/T_i = 1$ case. It is therefore not clear which theoretical model the radar experimenter should rely on in interpreting IS data collected from the daytime F region using the low-latitude stations such as Jicamarca and ALTAIR: collisionless models which can handle $T_e/T_i \neq 1$ [e.g., *Farley*, 1966] but become inaccurate as the aspect angle $\alpha \to 0$ or Woodman's collisional model for $T_e/T_i = 1$ which will give biased results at altitudes where $T_e/T_i > 1$? Using the nonanalytical model of *Sulzer and González* [1999] (or an empirical fit to *Sulzer and González* [1999] suggested by W. E. Swartz (private communication, 2005)) is not an option either since the model has not been developed in $\alpha \to 0$ limit where the radar beam is perpendicular to the magnetic field.

[4] The purpose of this note is to describe what we believe is the correct course of action when faced with the problem outlined above: Extract a core functional from Woodman's collisional model that effectively corresponds to a Gordeyev-type integral, and utilize the functional in the collisionless spectrum formula, e.g., of *Farley* [1966], derived for an arbitrary T_e/T_i. In what follows, we will explain why such substitutions should work and discuss the pertinent aspects of a resulting spectral model.

[5] In section 2 we overview the general framework of incoherent scatter spectral theories and the dependence of spectral formulae on Gordeyev-type integrals, one for each species of charge carrier in a given plasma. In section 3 we outline Woodman's spectral model derived for the $T_e/T_i = 1$ case and identify from the model what is effectively a collisional and magnetized Gordeyev integral. Since Gordeyev integrals are not a function of T_e/T_i, their use in the general spectral framework established by *Farley* [1966] and elsewhere is permissible. Section 3 includes the explicit formulae for the generalized collisional model, and section 4 includes a discussion of the model in terms of recent radar measurements.

2. General Framework of IS Spectral Models

[6] A number of independent approaches to derive the electron density spectrum of an equilibrium plasma with Maxwellian velocity distributions has led to identical results [e.g., *Farley et al.*, 1961; *Fejer*, 1961; *Hagfors and Brockelman*, 1971] which can be expressed as (we give here the singly ionized single-ion result since the multi-ion case is a straightforward extension)

$$\langle|n_e(\omega,\mathbf{k})|^2\rangle = \frac{|j\omega\epsilon_o + \sigma_i(\omega,\mathbf{k})|^2\langle|n_{te}(\omega,\mathbf{k})|^2\rangle}{|j\omega\epsilon_o + \sigma_e(\omega,\mathbf{k}) + \sigma_i(\omega,\mathbf{k})|^2} + \frac{|\sigma_e(\omega,\mathbf{k})|^2\langle|n_{ti}(\omega,\mathbf{k})|^2\rangle}{|j\omega\epsilon_o + \sigma_e(\omega,\mathbf{k}) + \sigma_i(\omega,\mathbf{k})|^2}, \quad (1)$$

where

$$\sigma_{e,i}(\omega,\mathbf{k}) = \frac{j\omega\epsilon_o}{k^2 h_{e,i}^2}\left[1 - j\theta_{e,i}J\left(\theta_{e,i}\right)\right] \quad (2)$$

is the longitudinal conductivity of electrons and ions,

$$\langle|n_{te,i}(\omega,\mathbf{k})|^2\rangle = \frac{2N_{e,i}\mathrm{Re}\{J\left(\theta_{e,i}\right)\}}{k\sqrt{2}C_{e,i}} \quad (3)$$

[1]Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Illinois, USA.

0148-0227/06/2005JA011546$09.00

represents thermally driven excitation spectra of the same species in the absence of collective interactions, and

$$\theta_{e,i} \equiv \frac{\omega - kV_{e,i}}{k\sqrt{2}C_{e,i}}. \quad (4)$$

Above, $h_{e,i} = C_{e,i}/\omega_{e,i}$ and $C_{e,i} = \sqrt{KT_{e,i}/m_{e,i}}$ are the Debye length and thermal speed of electrons and ions with mean densities $N_e = N_i = N$ and velocities $V_{e,i}$, respectively (along wave vector $\mathbf{k}$), while $J(\theta_{e,i})$ stands for Gordeyev integrals expressed in terms of normalized frequencies (4). In addition, $m_{e,i}$, $\mp_e$, and $T_{e,i}$ are particle masses, charges, and temperatures, respectively, K is Boltzmann constant, ϵ_o the free-space permittivity, and $\omega_{e,i} \equiv \sqrt{Ne^2/m_{e,i}\epsilon_o}$.

[7] The Gordeyev integrals $J(\theta_{e,i})$ depend on the physics of the plasma under consideration. They represent one-sided Fourier transforms (in normalized frequency units) of the normalized ACF of single particle echoes $\propto e^{j\mathbf{k}\cdot\mathbf{r}(t)}$ derived assuming the absence of collective interactions. For instance, in a magnetized but collisionless plasma

$$J(\theta) = \int_0^\infty dt\ \exp(-j\theta t) \cdot \exp\left\{-\left[\frac{t^2}{4}\sin^2\alpha + \frac{1}{\phi^2}\sin^2\left(\frac{\phi t}{2}\right)\cos^2\alpha\right]\right\}, \quad (5)$$

where

$$\phi_{e,i} \equiv \frac{\Omega_{e,i}}{k\sqrt{2}C_{e,i}} \quad (6)$$

corresponds to normalized gyro frequencies $\Omega_{e,i}$. Also, in (5) the second exponential in the integrand corresponds to the normalized ACF of single particle echoes in normalized time units.

[8] The appropriate Gordeyev integral for a given plasma can be identified either by deriving $\sigma_{e,i}(\omega, \mathbf{k})$ from plasma kinetic equations [e.g., *Farley*, 1966] or by deriving $\langle|n_{te,i}(\omega, \mathbf{k})|^2\rangle$ from a consideration of thermally driven dynamics of particles in the absence of collective interactions [e.g., *Hagfors and Brockelman*, 1971], and, in general, it is unnecessary to do both calculations in view of the Nyquist theorem, namely,

$$\frac{\omega^2}{k^2}e^2\langle|n_{te,i}(\omega,\mathbf{k})|^2\rangle = 2KT_{e,i}\mathrm{Re}\{\sigma_{e,i}(\omega,\mathbf{k})\}, \quad (7)$$

implied by (2) and (3) [e.g., *Farley*, 1966], and the well-known Kramers-Kronig relations between the real and imaginary parts of $\sigma_{e,i}(\omega, \mathbf{k})$ [e.g., *Yeh and Liu*, 1972].

[9] Once $J(\theta_{e,i})$ has been established for a given plasma by any appropriate means, the electron density spectrum (1) can be calculated as

$$\frac{\langle|n_e(\omega,\mathbf{k})|^2\rangle}{N} = \frac{|j(k^2h_e^2+\mu)+\mu\theta_iJ(\theta_i)|^2\ \frac{2\mathrm{Re}\{J(\theta_e)\}}{k\sqrt{2}C_e}}{|j(k^2h_e^2+1+\mu)+\theta_eJ(\theta_e)+\mu\theta_iJ(\theta_i)|^2} + \frac{|j+\theta_eJ(\theta_e)|^2\ \frac{2\mathrm{Re}\{J(\theta_i)\}}{k\sqrt{2}C_i}}{|j(k^2h_e^2+1+\mu)+\theta_eJ(\theta_e)+\mu\theta_iJ(\theta_i)|^2}, \quad (8)$$

where $\mu \equiv T_e/T_i$. For $T_e/T_i = 1$ and $kh_e \ll 1$, (8) simplifies as

$$\frac{\langle|n_e(\omega,\mathbf{k})|^2\rangle}{N} = \frac{|j+\theta_iJ(\theta_i)|^2\ \frac{2\mathrm{Re}\{J(\theta_e)\}}{k\sqrt{2}C_e}}{|j2+\theta_eJ(\theta_e)+\theta_iJ(\theta_i)|^2} + \frac{|j+\theta_eJ(\theta_e)|^2\ \frac{2\mathrm{Re}\{J(\theta_i)\}}{k\sqrt{2}C_i}}{|j2+\theta_eJ(\theta_e)+\theta_iJ(\theta_i)|^2}. \quad (9)$$

3. A Fokker-Planck Collisional Model for IS Spectrum at Arbitrary T_e/T_i

[10] *Woodman* [1967] obtains an explicit expression for (9) by using a Fokker-Planck like collision operator to model the electron and ion Coulomb collisions and an approach which is special to $T_e/T_i = 1$. While a generalization of Woodman's approach to an arbitrary T_e/T_i is nontrivial and has never been done, it is a straightforward exercise to match his result for $\langle|n_e(\omega, \mathbf{k})|^2\rangle/N$ with (9) and obtain an equivalent Gordeyev integral, namely [e.g., *Kudeki et al.*, 1999],

$$J(\theta) = \int_0^\infty dt\ e^{-j\theta t}\ \exp\left(-\frac{\psi t - 1 + e^{-\psi t}}{2\psi^2\sin^{-2}\alpha}\right) \cdot \exp\left(-\frac{\cos(2\gamma)+\psi t - e^{-\psi t}\cos(\phi t - 2\gamma)}{2(\psi^2+\phi^2)\cos^{-2}\alpha}\right), \quad (10)$$

where

$$\psi_{e,i} \equiv \frac{\nu_{e,i}}{k\sqrt{2}C_{e,i}} \quad (11)$$

denotes normalized collision frequencies $\nu_{e,i}$ and

$$\gamma_{e,i} \equiv \tan^{-1}\frac{\nu_{e,i}}{\Omega_{e,i}} = \tan^{-1}\frac{\psi_{e,i}}{\phi_{e,i}}. \quad (12)$$

The use of (10) within (8) amounts to a formal generalization of Woodman's collisional and magnetized spectral theory to the case of an arbitrary T_e/T_i. The generalized model consisting of (8) and (10) is free of singularities which appear in collisionless models in $\alpha \to 0$ limit. It can be conjectured that the same model will also be reached by deriving $\sigma_{e,i}(\omega, \mathbf{k})$ systematically from plasma kinetic equations using the same Fokker-Planck operator with the same approximations as those used by *Woodman* [1967]. However, such an exercise is unnecessary in view of the general framework of IS spectral theories outlined in section 2. We note here that the same framework was also utilized in a similar manner by *Sulzer and González* [1999] with their numerically simulated Gordeyev integrals at small but nonzero magnetic aspect angles.

4. Discussion and Conclusions

[11] The next question of importance is whether or not the generalized spectral model described by (8) and (10) is in agreement with physical reality. That is, to what extent the approximated Fokker-Planck collision operator utilized by

Woodman leads to an accurate spectral theory of incoherent scattered signals?

[12] A partial answer to the question, suggested by *Woodman* [2004], is that the model, used with Coulomb collision frequencies $\nu_{e,i}$ derived by *Spitzer* [1956], is expected to be accurate in the $\alpha \rightarrow 0$ limit, but the electron collision frequency ν_e needs to be amplified above its Spitzer value using a monotonically increasing function of angle α given in the paper (as equation (14)). The suggested correction to ν_e is explained to be a consequence of conflicting approximate treatments of the Fokker-Planck collision operator that can be justified at small and large α limits, and was in fact established by fitting Woodman's analytical spectral model (for $T_e/T_i = 1$, of course) to the spectra obtained from *Sulzer and González* [1999] model at small but finite α. While the fitting process does not cover the $\alpha \rightarrow 0$ limit, the reduction of the amplification factor to unity as $\alpha \rightarrow 0$ is consistent with the applicable approximation of the Fokker-Planck operator at $\alpha = 0$. Therefore the use of the generalized spectral model (8) and (10) at very small aspect angles seems to be at least as justified as the Fokker-Planck model for Coulomb collisions. At larger aspect angles the same model is subject to the accuracy of *Sulzer and González* [1999] model and the fitting procedure carried out by *Woodman* [2004]. Also, there are questions concerning a possible T_e/T_i and wavelength dependences of $\nu_e(\alpha)$ which are not addressed by *Woodman* [2004], but they can presumably be settled by repeating the fitting method using the generalized model (8) and (10).

[13] The ultimate evaluation of the accuracy of the evolving spectral models and their T_e/T_i and α dependence should be carried out using real radar data. In that regard, we have already applied a test to the generalized model (8) and (10) in terms of total power data collected with the ALTAIR UHF system in meridional *F* region scans (with time varying aspect angle α) across the geomagnetic field. The test, described by *Milla and Kudeki* [2006], gave reasonable electron density inversions that compared favorably with ionosonde results, and thus provided some confidence about the spectral and collision frequency models discussed above. However, the dependence of total radar cross section on Coulomb collisions is rather weak, and, consequently, the described test cannot be considered definitive. More stringent tests of the theories should be conducted using IS spectral data taken with radar beams scanned across the magnetic field lines with steerable systems such as ALTAIR or the future AMISR deployed at some low-latitude station, or using equivalent interferometric methods at Jicamarca.

[14] The IS spectral theory at small aspect angles still holds some mysteries and a potential for new and exciting results.

[15] **Acknowledgments.** We thank Ron Woodman and Koki Chau for many inspiring discussions we had on the theory and practice of ionospheric IS radars. This work has been funded at the University of Illinois by NASA and NSF grants NAG5-5368 and ATM-0215246, respectively.

[16] Arthur Richmond thanks Donald Farley and Jorge Chau for their assistance in evaluating manuscript 2005JA011546.

References

Farley, D. T. (1966), A theory of incoherent scattering of radio waves by a plasma: 4. The effect of unequal ion and electron temperatures, *J. Geophys. Res.*, *71*(17), 4091–4098.

Farley, D. T., J. P. Dougherty, and D. W. Barron (1961), A theory of incoherent scattering of radio waves by a plasma: 2. Scattering in a magnetic field, *Proc. R. Soc. London*, *263*(1313), 238–258.

Fejer, J. A. (1961), Scattering of radio waves by an ionized gas in thermal equilibrium in the presence of a uniform magnetic field, *Can. J. Phys.*, *39*, 716–740.

Hagfors, T., and R. A. Brockelman (1971), A theory of collision dominated electron density fluctuations in a plasma with applications to incoherent scattering, *Phys. Fluids*, *14*(6), 1143–1151.

Kudeki, E., S. Bhattacharya, and R. F. Woodman (1999), A new approach in incoherent scatter *F* region $E \times B$ drift measurements at Jicamarca, *J. Geophys. Res.*, *104*(A12), 28,145–28,162.

Milla, M., and E. Kudeki (2006), *F*-region electron density and T_e/T_i measurements using incoherent scatter power data collected at ALTAIR, *Ann. Geophys.*, in press.

Spitzer, L. (1956), *Physics of Fully Ionized Gases*, Wiley-Interscience, Hoboken, N. J.

Sulzer, M. P., and S. A. González (1999), The effect of electron Coulomb collisions on the incoherent scatter spectrum in the *F* region at Jicamarca, *J. Geophys. Res.*, *104*(A10), 22,535–22,551.

Woodman, R. F. (1967), Incoherent scattering of electromagnetic waves by a plasma, Ph.D. thesis, Harvard Univ., Cambridge, Mass.

Woodman, R. F. (2004), On a proper electron collision frequency for a Fokker-Planck collision model with Jicamarca applications, *J. Atmos. Sol. Terr. Phys.*, *66*(17), 1521–1541.

Yeh, K. C., and C. H. Liu (1972), *Theory of Ionospheric Waves*, Elsevier, New York.

E. Kudeki and M. Milla, Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, IL 61821, USA. (erhan@uiuc.edu; mmilla@uiuc.edu)