Notes on the complex autocorrelation function

R. Woodman, Sept. 1976
H. Kohl

In the analysis of narrowband signals, for instance radio signals, one normally uses a complex correlation function. A question is usually asked on the relationship of this complex function and the real functions of the real physical world. These notes are motivated by such question and discuss the advantages of such approach and its limitations.

Let f(t) be a real random, stationary and bandlimited (including quasisinusoidal) process. It is characterized by its real correlation function

$$R(\tau) \equiv \langle f(t)\, f(t+\tau)\rangle \qquad \text{(foot note 1)} \tag{1}$$

or its power spectrum

$$F(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau \,. \tag{2}$$

By bandlimited we mean processes for which $F(\omega) = 0$ for ω larger than a given frequency ω_{max}. Usually either $R(\tau)$ or $F(\omega)$ are sufficient to characterize the process, but it is usually more convenient, especially when the spectrum is relatively narrow (quasisinusoidal) and centered around a frequency ω_o, to characterize it alternatively in terms of two other functions $\rho(\tau)$ and $\varphi(\omega)$ which we will define in what follows.

Given $\omega_o < \omega_{max}$, we can always find a complex function $S(t) = a(t)+ib(t)$ such that

$$f(t) = \mathrm{Re}\left[S(t)\, e^{j\omega_o t}\right] \tag{3}$$

or

$$f(t) = a(t)\cos\omega_o t + b(t)\sin\omega_o t \tag{4}$$

Since we do not impose a condition on the Imaginary part of $S(t)\, e^{j\omega_o t}$, under general conditions the functions a(t), b(t) are not uniquely defined. But, this arbitrariness disappears if we impose the condition that a(t) and b(t) are bandlimited, i.e. if the spectrum of the real processes a(t) and b(t) are zero for ω larger than ω_o. Under this conditions a(t) and b(t) in 4) are

FN 1) ≡ implies definition, = equal sign.

unique (see Appendix) and hence also $S(t)$.

In terms of $S(t) = a(t)-jb(t)$ we can write (see Appendix)

$$R(\tau) = \frac{1}{2}\mathrm{Re}\left[<S(t)\overset{*}{S}(t+\tau)> e^{-j\omega_0\tau}\right]. \tag{5}$$

Let us define the complex autocorrelation function

$$\rho(\tau) \equiv < S(t)S^*(t+\tau)>. \tag{6}$$

Because of the real and stationary character of $f(t)$ it can be shown (Appendix) that:

$$\rho(\tau) = \rho^*(-\tau) \quad \text{i.e. } \rho(\tau) \text{ has Hermitian symmetry.} \tag{7}$$

We can also define a function

$$\varphi(\omega) \equiv \int_{-\infty}^{\infty} \rho(\tau)\; e^{-j\omega_0\tau}\, d\tau \tag{8}$$

which we will call the "converted spectrum" of $f(t)$.

Because of the Hermitian symmetry of $\rho(\tau)$, $\varphi(\omega)$ is real. The band limited condition on $a(t)$ and $b(t)$ implies that:

$$\varphi(\omega) = 0 \text{ for } |\omega| > \omega_0 \tag{9}$$

In terms of $\rho(\tau)$ and $\varphi(\omega)$ we can write

$$R(\tau) = \mathrm{Re}\left[\rho(\tau)\; e^{-j\omega_0\tau}\right] \tag{10}$$

and

$$F(\omega) = \frac{1}{4}\int \rho(\tau)\; e^{-j(\omega+\omega_0)}\; dt + \frac{1}{4}\int \rho^*(\tau) e^{j(\omega_0-\omega)\tau}\, d\tau,$$

or (11)

$$F(\omega) = \frac{1}{4}\,\varphi(\omega+\omega_0) + \frac{1}{4}\,\varphi(\omega_0-\omega),$$

and using the "band limited" conditions we can write

$$F(\omega) = \frac{1}{2}\,\varphi(\omega+\omega_0) \text{ for } \omega > 0$$

$$= \frac{1}{2}\,\varphi(\omega_0-\omega) \text{ for } \omega < 0 \tag{12}$$

Discussion

Since ω_0 is known (given) then $R(\tau)$ is easily derived from $\rho(\tau)$ (Eq.(10)). Also $F(\omega)$ is easily derived from $\varphi(\omega)$ (Eq. (12)). Eq. 12 implies that $F(\omega)$ is completely determined by $\varphi(\omega)$, which is usually narrow, in that $F(\omega)$ equals $\varphi(\omega)$ displaced by an amount ω_0 from its origin plus its symmetric counterpart. These interrelationships are so direct that one need not go back to the real functions $R(\tau)$ and $F(\omega)$ for a physical interpretation of the process.

There is a practical advantage in using and obtaining $\rho(\tau)$ and or $\varphi(\omega)$ instead of the real $R(\tau)$ and $F(\omega)$, which lies in the fact that $\rho(\tau)$ is much less structured than $R(\tau)$ and therefore needs less points for its practical determination and in that $\varphi(\omega)$ has non zero values only close to the origin and can easily be plotted.(See Figure)

The terms $a(t)$ and $b(t)$ can be obtained experimentally in the real physical world. From the definition 4) we have

$$f(t) = \frac{1}{2}\left[S(t)e^{j\omega_0 t} + S^*(t)e^{-j\omega_0 t}\right] \tag{13}$$

Multiplying by $e^{-j\omega_0 t}$ and solving for $S(t)$ we get

$$S(t) = 2f(t)\,e^{-j\omega_0 t} - S^*(t)e^{-j2\omega_0 t}. \tag{14}$$

Since $S(t)$ has a spectrum within 0 and ω_0 the second term can be filtered out by letting $S(t)$ as given in 13] go through a filter sufficiently narrow to let the first term go through and filter out $S^*(\omega)\ e^{-j2\omega_0}$.
Thus

$$S(t) = \text{Filter}\left[f(t)e^{-j\omega_0 t}\right].$$

In practice $S(t)$ is obtained by multiplying $f(t)$ by $\cos\omega_0 t$ and filter to obtain $a(t)$ and by multiplying it by $\sin\omega_0 t$ and filter to obtain the imaginary part $b(t)$. Notice that although $-jb(t)$ is the imaginary part of $S(t)$, $b(t)$ itself is real and actually exist in the physical world. The autocorrelation $\rho(\tau)$ is then obtained directly by evaluating

$$\rho(\tau) = \langle a(t)a(t+\tau)\rangle + \langle b(t)b(t+\tau)\rangle + j\langle a(t)b(t+\tau)\rangle - j\langle b(t)a(t+\tau)\rangle.$$

The sampling rates are determined by the bandwidth of $\varphi(\omega)$ that is by the relatively slow characteristic times of $\rho(\tau)$. This is an additional advantage since otherwise one would have to sample at twice the highest frequencies of $f(t)$, i.e. at frequencies at least of the order of $2\omega_0$ (as high as $4\omega_0$).

Notice that the approach is valid for all signals with a spectrum within $\pm 2\omega_0$. They need not be narrow around ω_0, although this is usually the case and it is in this case that the approach presents practical advantages. Also there would be some technical problem with the filtering, if $F(\omega) \neq 0$ at the low frequencies as well.

Appendix

Here we shall show with the help of the Nyquist sampling theorem that under the bandlimiting assumptions $a(t)$ and $b(t)$ in equation (4) of the main text are uniquely determined. We shall also derive the stationarity of $S(t)$ and its Hermitian properties.

Let $f(t)$ be real and statistically stationary,

$$\text{i.e.}\quad R(\tau) = \langle f(t)f(t+\tau)\rangle = \langle f(t+T)f(t+T+\tau)\rangle \text{ for all } T\text{'s}$$

and bandlimited, i.e.

$$F(\omega) = 0 \quad \text{for } \omega > 2\omega_0$$

we can always write

$$f(t) = a(t)\cos\omega_0 t + b(t)\sin\omega_0 t$$

From Nyquist sampling theorem we know that $f(t)$ (and also $R(\tau)$) is fully defined if known at times t_i (or τ_i) given by

$$t_i = 0,\ T/4,\ 2T/4,\ \ldots\ldots \qquad \text{where } T = 2\pi/\omega_0$$

At these specific times

$$a(t_j) = f(t_j) \qquad \text{for } j = 0,2,4,6 \ldots\ldots.$$
$$b(t_i) = f(t_i) \qquad \text{for } i = 1,3,5,7 \ldots\ldots. \qquad (2)$$

Given $a(t_j)$, $b(t_i)$ they determine a unique set of functions $a(t)$, $b(t)$ if we impose the condition they take the values $a(t_j)$ and $b(t_i)$ at t_j, t_i and further if we assume or force them to be band limited, that is if their spectrum $F_a(\omega) = F_b(\omega) = 0$ for $|\omega| > \omega_0$. Therefore for a given function $f(t)$ and a frequency ω_0 there is a unique set of functions $a(t)$, $b(t)$ satisfying 1). In addition we have from 1) that

$$R(\tau) = \frac{1}{4}\left< \left[S(t)\, e^{j\omega_0 t} + S^*(t)e^{-j\omega_0 t}\right]\left[S(t+\tau)e^{j\omega_0(t+\tau)} + S^*(t+\tau)e^{-j\omega_0(t+\tau)}\right]\right>$$

or

$$R(\tau) = \frac{1}{2}\mathrm{Re}\left[<S(t)S^*(t+\tau)>e^{j\omega_0\tau}\right] + \frac{1}{2}\mathrm{Re}\left[<S(t)S(t+\tau)>e^{j\omega_0(2t+\tau)}\right]$$

where

$$S(t) = a(t)-jb(t).$$

Since $f(t)$ is stationary, then $R(\tau)$ is independent of t. This implies that $e^{j\omega_0(2t+\tau)}$ $\rho(\tau) \equiv <S(t)S^*(t+\tau)>$ is independent of t(stationary $S(t)$) and that $<S(t)S(t+\tau)>=0$ for all t's and τ's, i.e.:

$$\left[<a(t)b(t+\tau)>-<b(t)a(t+\tau)>\right]\cos\omega_0(2t+\tau) - i\left[<a(t)b(t+\tau)>+<b(t)a(t+\tau)>\right]=0.$$

$\sin\omega_0(2t+\tau)$

This in turn implies that

$$<a(t)a(t+\tau)>=<b(t)b(t+\tau)> \qquad \text{for all t's} \qquad (3)$$
$$<a(t)b(t+\tau)>=-<b(t)a(t+\tau)>$$

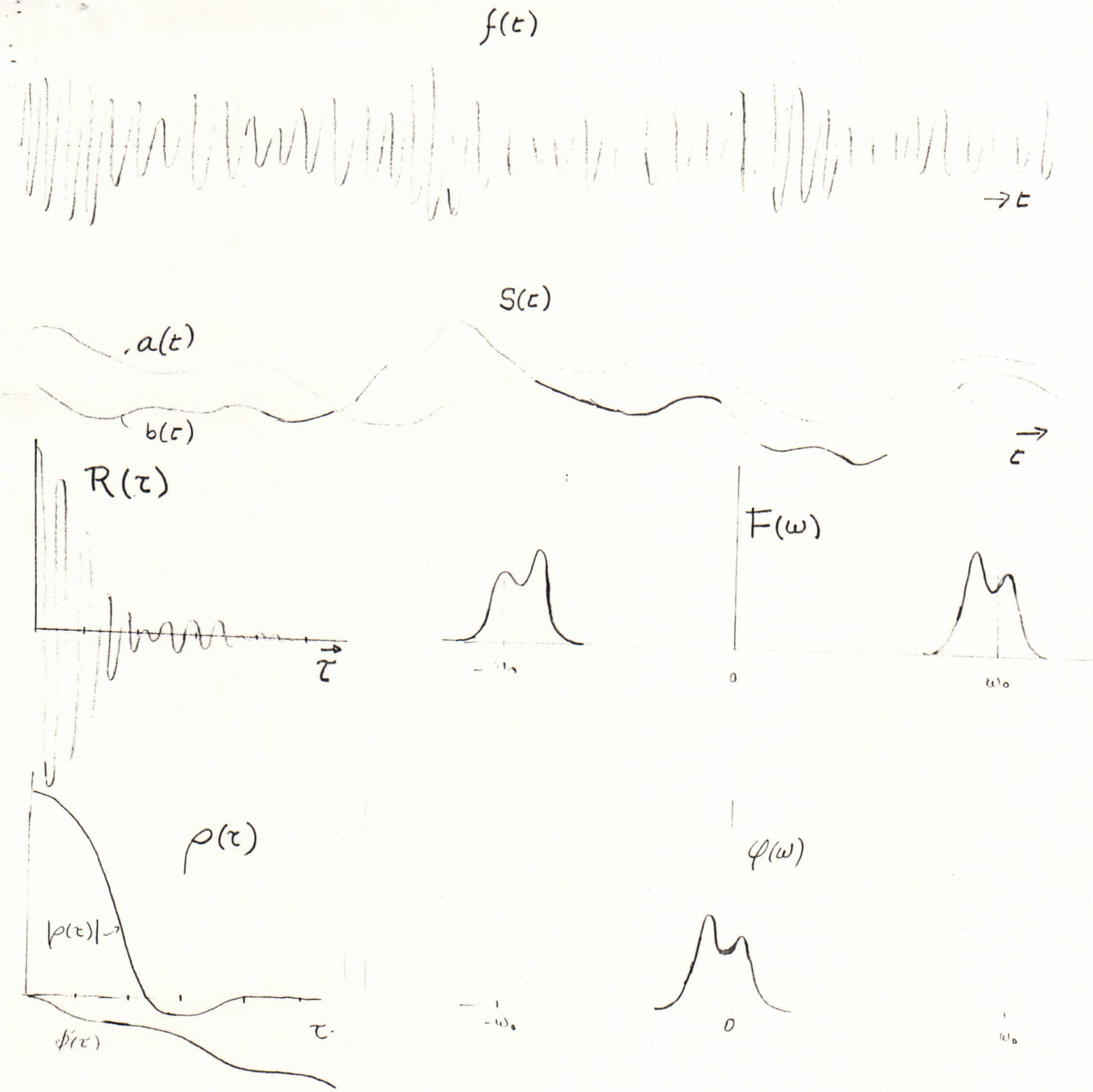
and that

$$\mathrm{Re}\ \rho(\tau) = \mathrm{Re}\ \rho(-\tau),$$
$$\mathrm{Im}\ \rho(\tau) = -\mathrm{Im}\ \rho(-\tau).$$

Consequently $\rho(\tau)$ has Hermitian symmetry.
i.e.

$$\rho(\tau) = \rho^*(-\tau). \qquad (4)$$

Noti-e that equations 2) present an alternative solution in obtaining $a(t)$ and $b(t)$ to the one presented in the main text. One can obtain $a(t)$ and $b(t)$ directly by sampling a pair of values at a $t/4$ interval. If the bandwidth of the process is much smaller than ω_0, let us say $\varphi(\omega)$ is such that $\varphi(\omega) = 0$ for $\omega > \Omega$ where, $\Omega \ll \omega_0$; then, the pair of sampling need to be taken only once every $T'/2$, where $T' = 2\pi/\Omega$.



Schematic plots of the different functions mentioned in the text.