Middle Atmosphere Program

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Chapter 4

Statistical Characteristics of MST Radar Echoes and its Interpretation[1]

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Introduction

As we shall see later, radar backscattering is produced by fluctuations in the refractive index of the illuminated medium with scale sizes equal to 1/2 the wave length of the electromagnetic probing wave. The fluctuations are a random process, and so are, consequently, the signals received by the radar. Both have to be characterized statistically. The power of the technique is based on the fact that the statistical parameters that define the signal received are related to the statistical parameters of the medium. This allows us to remote–sense the medium from the ground.

It is important, then, in order to understand the technique, to know the statistical ways of characterizing 1) the fluctuations in refractive index and 2) signals received. The second may be familiar to many of you. The first may not. The second is easier to understand since it is a one dimensional process (time). The first is harder, since involves processes in four dimensions, 3 in space, and 1 in time; on the other hand, it uses extensions of concepts developed originally for one dimension, and should present no difficulties if these one dimensional concepts are understood.

Fluctuations in Index of refraction come about mainly as a consequence of atmospheric turbulence. If we are going to use these fluctuations to study the atmosphere, it is important, in order to interpret the signals received, that we understand some of the fundamental concepts related to atmospheric turbulence.

Because of above reasons, we have decided before entering on the main subject of our lecture, that of the characterization of radar echoes and its interpretation, to review some fundamental concepts in random process statistics and in atmospheric turbulence.

Some statistical concepts

There are two concepts of fundamental importance which should be reviewed: Autocorrelation Function and Frequency Power Spectrum. They are interrelated. One can be defined in terms of the other. Mathematically it is much simpler to define the first, although many find easier to grasp the physical significance of the second.

Given a time series, either as a sequence of numbers in time $s_1 s_2 s_3 .. s_i ..$, or as a random function of time, s(t), (We will use s(t) for both cases for convenience, unless we want to stress the discrete nature of a sequence), its autocorrelation function is defined as:

$$\rho(\tau) = E[s(t), s(t+\tau)] \qquad (1)$$

where E[ ] stands for the expectation of its argument. Good estimators of this expectation are:

$$\rho'(\tau) = \overline{s(t)s(t+\tau)}^{T}, \qquad (2)$$

if the process is stationary ,or, under more general conditions,

$$\rho'(\tau) = \langle s(t)s(t+\tau)\rangle_n . \qquad (3)$$

The overbar stands for a time average of duration T, and the brackets stand for averaging over n identical experiments or observations.

The second estimator allows us to evaluate correlation functions even in the case the process is not stationary. When the process is not stationary, we should write $\rho(\tau; t)$, to stress the dependence on t, the time at which the correlation is evaluated.

Let us see what a correlation function means physically. Let us take equation (2) as a good definition (it is, for all practical purposes, if the time T taken for the average is long enough). Fig. 1 show a sample function of the

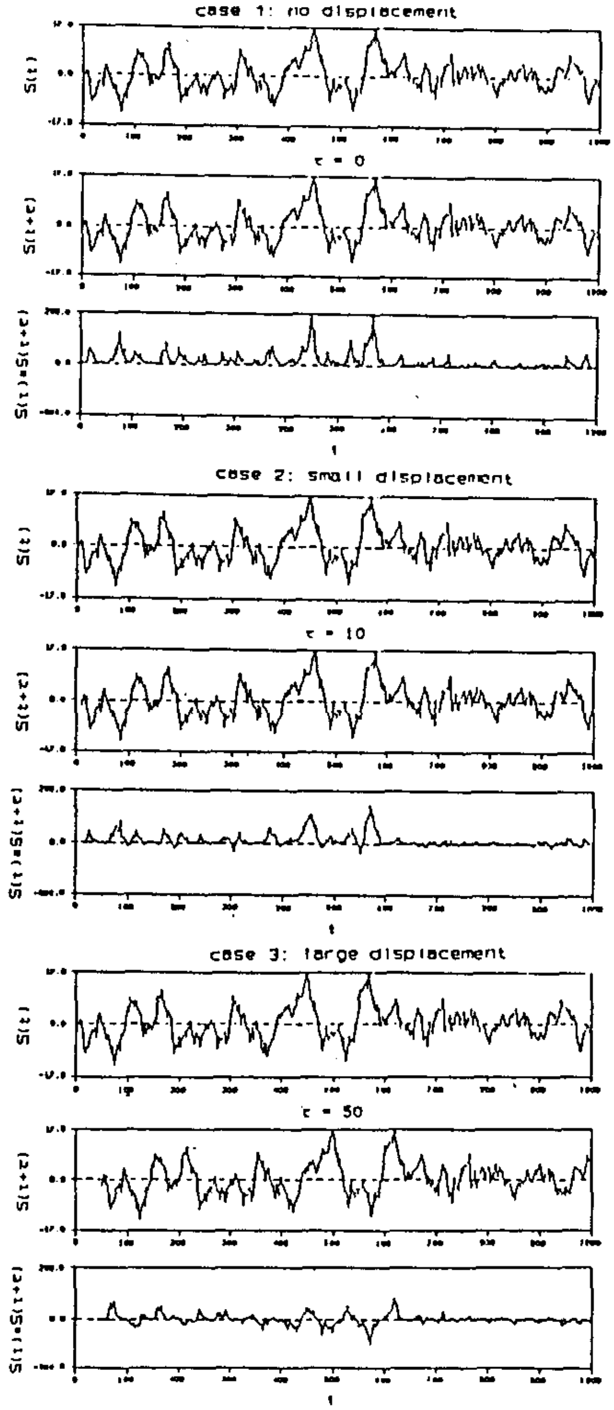

Figure 1 – Three realizations of the $s(t)s(t+\tau)$ process illustrating its eventual contribution to $\rho'(\tau) = \overline{s(t)s(t+\tau)}^{T}$, for $\tau = 0$, small and large.

random functions $s(t)$,$s(t+\tau)$ and $s(t)s(t+\tau)$ for three displacements, $\tau=0$, $\tau=$ "small" and $\tau=$ "large". When $\tau=0$, we get $s(t)^2$ for the product function, the integral of which corresponds to an estimation of the power of the process, which we use as a reference.

If we increase t by small amount, $s(t)s(t+\tau)$ does not vary much from the $\tau=0$ case, and the integral is slightly smaller than the power . If τ is large enough, it is equally probable for the product to be positive or negative, and the integral is zero.

But, what is small enough and what is large enough? The answer is given by autocorrelation function itself. Note that, between the two τ's depicted in figure 1, there should be a τ, $\tau_{1/2}$, at which the correlation is equal to $0.5\rho(0)$ and that the correlation function decays from its maximum value to zero in a characteristic time, τ_c . This characteristic time or, alternatively, one derived from the normalized second moment of $\rho(\tau)$, has a ready interpretation and gives us an idea of how fast the process varies. It can be centuries (changes in the global temperature of the earth) or hours (changes in the ambient temperature) or, seconds (changes in the punctual temperature of a turbulent process) or any other time scale. This is the most usual interpretation given to the correlation function. There is more information , of course, besides the power and the characteristic time of the process in the functional shape of the correlation function; for instance, if the shape is oscillatory it tells us that the process is quasi-sinusoidal with a period given by the period of the oscillations. Nevertheless, in many cases, it is sufficient to give only this simple interpretation.

Power spectrum – when defined carefully (e.g. Papoulis,1965)– is defined as the Fourier Transform of $\rho(\tau)$, namely

$$F(\omega) = 1/2\pi\int_{-\infty}^{\infty}\rho(\tau)\ \exp(-j\omega\tau)\ d\tau \tag{4}$$

This is is a modern definition. The earlier definition and, in any case, a good interpretative way of looking at it, is that the power spectrum, $F(\omega)$, measures the power density of a process at different frequencies. This means that, if the process is fed to a bank of filters centered at frequency ω_i, The average power of each filter would be proportional to $F(\omega_i)$, where ω_i is the center frequency of the filter. There are many estimators of $F(\omega)$ which conform to this definition. For instance

$$F'(\omega) = \left\langle \left| 1/T \int_{-\infty}^{\infty} f(t) \exp(-j\omega t)\, dt \right|^2 \right\rangle_n \tag{5}$$

This is equivalent to getting the Fourier transform of a subset of the sequence, obtain its power (square it) and average many sub–sequences.

Extension to 3–D and time processes

A good example of a three space dimensions and time random process is the temperature or the velocity of a boiling pan of water, or any other turbulent process. These processes are also characterized by its autocorrelation function, $\rho(\mathbf{r},t)$. It is defined in a fashion similar to its one dimensional case. For instance, if we take n to stand for the deviations in density, or the refractive index of a medium , its autocorrelation function is defined as :

$$\rho(\mathbf{r},\tau) = E[n(\mathbf{x},t)n(\mathbf{x}+\mathbf{r},t+\tau)] \tag{6}$$

That is, it is the expectation (in practice,the average) of the product of the density at point $\mathbf{x}$ at time t , multiplied by the density at a point displaced $\mathbf{r}$ from $\mathbf{x}$, at a time τ units later. If the medium is stationary and homogeneous ρ does not depend on $\mathbf{x}$ or t. Otherwise, we should write $\rho(\mathbf{r},\tau;\mathbf{x},t)$, since the autocorrelation would be different if measured in a different place or at different time.

As in the case of one dimension, there is a characteristic length, r_c , and a characteristic time, τ_c, much beyond which the autocorrelation is small or zero. If the medium is isotropic, the characteristic length is the same , regardless of the direction of the displacement, $\mathbf{r}$. In this case we can use the magnitude , r, instead of the vector , $\mathbf{r}$. If the medium is anisotropic, there can be as many as three characteristic lengths, one in each major axis direction.

As in the case of onedimension, the characteristic time gives us an idea of how long we have to wait before the three dimensional structure of a sample process changes significantly. Similarly, the characteristic scale gives us an idea of how far we have to move from a specific point, from which we have taken a snap shot at the process for a second snap shot, taken at the same instant, to differ significantly and yet show some resemblance. The directions of displacement should preferably be taken along the three major axis of the correlation function.

To envision the meaning of statistical anisotropy, let us consider the two dimensional case of the vertical displacement of the surface of a choppy ocean produced by a wind of constant direction. Here, there would be a tendency for the waves, or even swell, to form in with preference in one direction, that of the wind. If we displace ourselves along the crests of the waves, we have to move much further for observational snapshots to look different than if we displace ourselves along the direction of propagation of the waves (direction of the wind). The characteristic lengths in this case are different, being shorter along the direction of the wind.

Again, for the purpose of an introductory interpretation, we have talked about a single parameter per dimension. This is over simplified. One or few parameters does not replace the whole correlation function unless we accompany it with knowledge of its functional shape (e.g. Gaussian, Lorentian, sinusoidal, exponential,etc.), or by a sufficient number of evaluated points.

There is also a counterpart in 3–D processes to the concept of frequency power spectrum. In this case we speak of wave–number–vector (extension of wave–number) power spectrum, or **k**–spectrum.. In an analogous fashion, we define it as the 3–D spatial Fourier transform of the space–time autocorrelation function, $\rho(\mathbf{r},\tau)$, specifically,

$$\phi(\mathbf{k}) = 1/(2\pi)^3 \int_{-\infty}^{\infty} \rho(\mathbf{r},0) \exp(-j\mathbf{k}.\mathbf{r})\, d^3\mathbf{r} \qquad (7)$$

Note that we have set τ equal to zero . Therefore in this definition, we are performing the displacements in space at the same instant of time, i.e. no time dynamics is included. We could also have used $\rho(\mathbf{r})$ as a symbol for the same concept. Again, its interpretation is similar to the frequency power spectrum. We can interpret $\phi(\mathbf{k})$ as a function which describes the "power" density of the different wave number components of the process. We imagine the process to result from the (Fourier) superposition of different spatial waves with different directions and wavelengths , each with a power (amplitude squared) given by $\phi(\mathbf{k})$.

There is an important concept in talking about the directional scattering properties of a medium. One talks about the aspect sensitivity of the scatterers. It is a consequence of the anisotropic character of the $\phi(\mathbf{k})$ which characterizes anisotropic turbulent fluctuations. This anisotropy is sometimes better perceived from the shape of the autocorrelation function, $\rho(\mathbf{r})$. In this regard it should be kept in mind that, in any Fourier pair, like $\phi(\mathbf{k})$ and $\rho(\mathbf{r})$, wide functions transform into narrow functions and viceversa. This means that if we have a horizontal, pancake–like spatial autocorrelation function, it transforms into a vertical pencil–like **k**–spectrum.

We can relax, above, the restriction for τ to be zero. We would obtain a function, $\phi(\mathbf{k},\tau)$, which associates certain dynamics to each spatial wave component. Each component will have a characteristic time associated to its life time. This does not mean that the process no longer has power at that particular wave–number vector, but rather that wave component is completely independent of the one observed a few characteristic times ,τ_c, ago.

To further complicate matters, we can perform an additional Fourier transformation in time on ρ . We would obtain

$$\Phi(\mathbf{k},\omega) = (1/2\pi)^4 \int_{-\infty}^{\infty} \rho(\mathbf{r}, \tau) \exp(-j\mathbf{k}.\mathbf{r}-j\omega\tau)\, d^3r d\tau \ . \qquad (8)$$

In this case the dynamics of the process, for each wave–number vector, **k**, is represented by a superposition of temporal oscillations with frequency ω, and power density $\Phi(\mathbf{k},\omega)$.

We are not presenting this concepts for purely academical reasons. As we shall see later, the signal statistics of the echoes received in a MST radar are directly related to the spectrum $\phi_n(\mathbf{k}_s,\tau)$ (or $\Phi_n(\mathbf{k}_s,\omega)$) which characterizes the density fluctuations of the medium. Although here, $\mathbf{k}_s$ is no longer a variable but an specific wave–number vector determined by the frequency and geometry of the radar. We should, then, be familiar not only with the mathematical definition of these concepts, but with their physical significance as well. Only then we can attribute physical significance to the results of a MST radar experiment.

We have used the terms stationary and homogeneous. In the theory of random process, they are defined as follows. A process is said to be stationary, if the expectation of any function of its value, or values (for instance E[s(t)] and

$E[s(t)s(t+\tau)]$), is independent of the time of the sample function taken. It is said to be homogeneous, if the expectation is independent of where the values of the sample function are taken.

In the exact context of this definition, the time and physical space have to be infinite in extend. In practice one uses the concept of quasi-stationarity and quasi-homogeneity, in which the "any time" or "any point" implicit in the strict definition is replaced by finite intervals of time and finite regions of space, sufficiently large as to contain a large number of characteristic times and length scales. The assumption of stationarity or homogeneity is considered to be valid if they hold within a particular observation time or region.

For further reading see Papoulis (1965) and Tatarsky (1961).

Some turbulence concepts.

The MST radar depends on turbulence to obtain echoes from the clear atmosphere. It uses turbulence as a tracer of the dynamics of the background atmosphere. Also, since the statistical parameters of the received signal depend on the statistical parameters of the refractive index fluctuations –produced by turbulence–, the radar can also be used to study the turbulence process proper. It is important, then, to understand some of the basics of atmospheric turbulence.

We would like to underline "basics" since turbulence theory is a difficult subject. In fact, as a consequence of its highly non linear behavior, and in spite of all the advances in its mathematical description, we are still not able to predict its behavior, even in a statistical sense.

The meaning of turbulence varies from a general dictionary type definition to controversial and more limited definitions. For us, it suffices to define it as the state of a fluid in which the velocity field is rotational and random in three dimensions and time.

Although some atmospheric physicists envision the existence of two (space) dimensional turbulence in the atmosphere, we will use the term only in a three dimensional context. We are interested in 3–D turbulence with length scales no larger than about a few hundreds of meters in the stratosphere and stable troposphere (non–convective) and a few hundred to slightly above a thousand meters in the mesosphere. We are also occasionally interested in the small scales (meters to hundred of meters) as well as the larger (kilometers) scales of tropospheric convectional turbulence.

For turbulence to exist we need a fluctuating velocity field. Radars, on the other hand, are sensitive to fluctuations in refractive index or, equivalently, fluctuations in density or temperature at constant pressure. Fortunately, in most cases, velocity fluctuations bring about density fluctuations, although this is not always the case.

If we consider a non stratified atmosphere (no gravity) at constant pressure, veldcity fluctuations would not produce density fluctuations. Different parcels of air would interchange positions, but since they have the same density, no fluctuations would be produced. But, if a gradient of density exist, for any reason, then, regions of higher density would be brought to regions with lower density and viceversa, producing fluctuations in density and hence in refractive index.

If we steer pure water, for instance, we could not perceive optically any change, but if we mix it with clear syrup, it would produce a whitish fluid (while the emulsion last) as a consequence of the light scattering the small scale fluctuations in refractive index are capable to produce.

Mixing in a gravitational stratified atmosphere is slightly more complicated. We have to introduce in this case the concept of "potential density" and "potential temperature".

Let us consider a medium with a constant temperature profile. Under the influence of gravity it would have a density like $n = \exp(-z/H)$. If we interchange two parcels of different altitudes adiabatically and in pressure equilibrium, we would cool by decompression the parcel moving up into a lower pressure, and heat the parcel moving down into higher pressure. So, if we steer locally an atmosphere with a constant temperature profile, we end up with fluctuations in temperature, apparently contradicting ourselves. It is more convenient – conceptually and mathematically – to characterize, instead, the state of the medium by the temperature it would have if it were to be brought to sea level adiabatically. This "temperature" is called potential temperature. It is a conserved property of the medium, i.e. it does not change as it is moved adiabatically to other altitudes. In the language of turbulence theory it is said that it behaves as a passive scalar. We can define a potential density in a similar fashion.

For turbulence to produce fluctuations in density or temperature we need a gradient in potential density. Constant potential density backgrounds do not produce fluctuations. When an atmosphere has such profile, we say that the (actual) temperature has an adiabatic lapse rate (about 1 every 100 meters). The stratosphere has either a constant or positive gradient temperature profile, hence it deviates more from an adiabatic lapse rate than the troposphere. It is potentially capable, then, to produce larger fluctuations for the same mixed layer thickness than the troposphere.

In the mesosphere the refractive index is produced by the density of free electrons. The gradient of both potential and real electron density gradient is positive and hence capable of producing refractive index fluctuations when mixed by turbulence.

Assuming an initial gradient in a passive scalar, one can derive (e.g. Tatarsky,1961) a quantitative formula relating the standard deviation of the scalar (like potential temperature or potential refractive index) in terms of the original gradient and the depth of the turbulent mixing layer thickness. Assuming further a Kolmogorov power spectrum density law (see below), that is a dependence of ϕ on k of the form $k^{-11/3}$. He derived and expression for the standard deviation of the fluctuations of the form

$$\phi_n(k)= a\ 0.033L_o^{4/3}\ (\mathrm{grad}\ n)^2\ k^{-11/3} \qquad (9)$$

As expected the fluctuation density at any wavelength is directly proportional to positive powers of the original gradient and the scale of the largest mixing eddy, L_o. "a" is a constant of order unity.

We can also estimate roughly the variance on the velocity field in the following way. If we mix a (stable) gradient in potential density we produce work, since we are moving up potentially heavier and down potentially lighter parcels of air. We need then an energy source. This source comes from shear. Without shear, there is no source and no turbulence. The original shear after turbulence is reduced to very low value due to turbulent viscosity. The excess of kinetic energy resultant from the difference in velocity of the originally shear profile and the new constant velocity profile (see fig. 2) has to go into potential energy, result of the work we mention earlier, and the random turbulent kinetic energy. It we assume equipartition of the energy derived from the shear into 4 parts, 3 for the 3 different orthogonal components of the turbulent velocity ($\langle u'^2\rangle$, $\langle v'^2\rangle$, $\langle w'^2\rangle$) and one for the potential energy, and we further assume a normal distribution of velocities, we can derive that the variance of any of the velocity components would be approximately (Woodman and Guillen, 1974; Sato and Woodman, 1982):

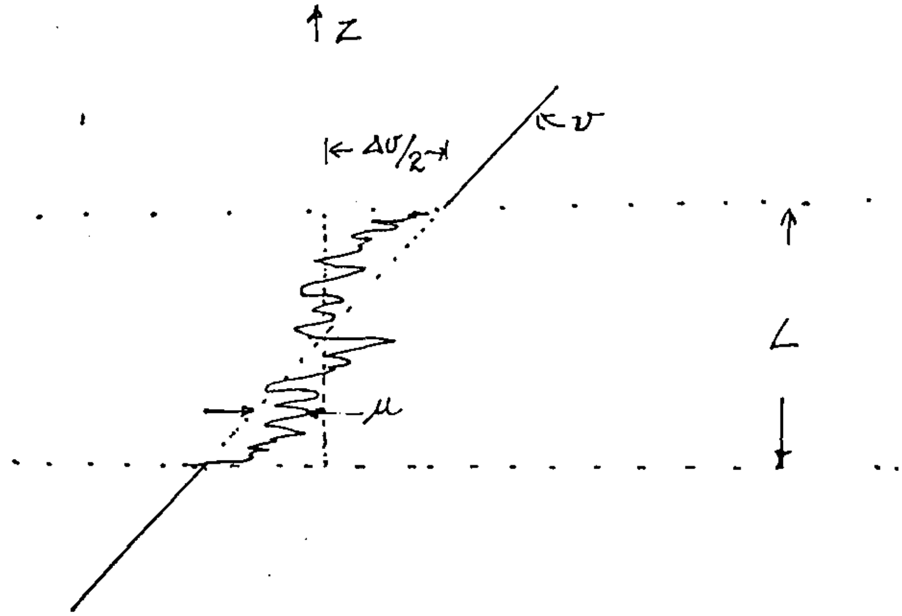

Figure 2 – Schematic profile of the turbulent fluctuating component, u, and its relationship to Δv, the shear component that is randomized by turbulence

$$\langle u'^2 \rangle = 1/48(\Delta v)^2 \tag{10}$$

where Δv is the difference in velocities between the top and bottom of the layered region that went turbulent i.e.

$$\langle u'^2 \rangle = 1/48(\ L_o\, dv/dz\ )^2 \tag{11}$$

A normal distribution of velocities is a fair assumption, since it parcel of fluid is influenced by the superposition in space and time (velocity is the integral of force) of many independent forces and the limit theorem applies. This is an important additional statistical property of the medium with consequences in the shape of the correlation and spectrum of the signal.

A related subject to that of equations (10) and (11) is that of Richardson's criteria for stability. It says that a layer is unstable if

$$Ri \equiv (g\ d/dz\ \ln\theta)/\ (dv/dz)^2 \leq 1/4 \tag{12}$$

The criteria can be interpreted as a condition for turbulence to be energetically possible, namely the available kinetic energy in the shear has to be 4 times larger than the gain in potential energy after the mixing. This is in agrement with above arguments.

Some of these criteria can be used to extract hidden information from MST radar experiments, information that on first thought should not be available. Woodman and Guillen, for instance, using above relations, assuming that the original shears are marginally unstable, and from the measured values of the spectral width, deduced that the turbulent layers in the stratosphere were of the order of 50 meters, even though the resolution of the instrument was 5 km. Sato and Woodman have later validated this arguments by measuring $\langle u'^2 \rangle$ and L_o with the 150 meter resolution 430 MHz radar at Arecibo.

Richardson's criteria tells us that turbulence is energetically possible, but it does not tell us how it comes about. We need an unstable process that would make small disturbances grow and eventually brake down into the non-linear regime that we call turbulence. One such a process is the Kelvin-Helmholtz instability. The process is analogous to the way wind, blowing on the ocean surface, peaks a particular wave, that which has a phase velocity equal to the wind velocity, and make it grow until it breaks down. In the atmosphere shear effectively produces a wind that blows with respect to the denser fluid underneath, it peaks a particular gravity (buoyancy) wave, and makes it grow until eventually brake into a billow and this in turn into smaller scale turbulence. The phenomena is confined to the layers within which the process is energetically possible, i.e. were Richardson's criteria is satisfied.

Turbulence is also possible without shear, if the numerator in equation (12), that is if the gradient in potential temperature, is also zero or negative. We then say that the atmosphere is statically unstable. We effectively have a heavier fluid resting on top of a lighter one, a condition that is definitely unstable (Raleigh–Taylor instability).

Both processes mentioned above, Kelvin–Helmholtz and Raleigh–Taylor instabilies, can come about in the atmosphere as a consequence of large amplitude gravity and lower frequency waves in the atmosphere. These waves have a velocity field which is transverse to their **k** . Their **k**–vector is almost vertical. It is then possible, as the waves grow in amplitude with height, to produce almost horizontal shears that satisfy Richardson's criteria. The slight tilt of the velocity field of the wave is also capable to lift regions of higher (potential) density above regions of lower density, making them statically unstable.

An often quoted and very important conclusion that has come out of turbulence theory is Kolgomorov's wave–number spectrum. It says that within a given range of wave–number values the wave–number power spectra is of the form

$$k^2\phi(\mathbf{k}) \propto k^{-5/3} \tag{13}$$

We have place the k^2 factor on the left hand side to conform with the –5/3 power law which is often quoted in the literature. The difference comes from the use of what is referred as the one dimensional (in three dimensions) spectrum, in which the Fourier transformation from **r**–space to **k**-space is performed by transforming in one dimension integrating along the magnitude of r.

The range within which this law is valid is called the "inertial subrange". The relationship can be derived on pure dimensional arguments with the assumption that for scales smaller than the primary energy containing scales, but large enough so that molecular viscosity does not play a role, there should be a dimensionless relationship between eddies of different sizes and that they should be isotropic. The law brakes down at dimensions close to the largest eddy possible, and on the other end, at small dimensions where the inertial forces are comparable to the ones produced by molecular viscosity , i.e. at scales where molecular viscosity becomes important in extracting energy (into thermal) from the eddies. Within the inertial subrange, kinetic energy is cascaded from the larger to the neighboring smaller eddies.

Kolmogorov's law is isotropic and valid for non stratified media. In the case of the gravity stratified atmosphere, Kolgomorov's law is valid for the smaller scales, where potential energy is smaller than kinetic energy. On the larger scale it fails before it reaches the largest scales. The region between the "outer scale " and the inertial subrange, where potential energy is significant is referred to as the "buoyancy subrange". Not only the $-5/3$ power law fails; isotropy is no longer true, gravity, and the unstable phenomena responsible for the larger eddies, have preferred directions which spoil the isotropic symmetry.

The turbulent state of a fluid is often specified by the outer scale, i.e. the size of the largest eddies, and the energy dissipation rate, ε (e.g. Hocking,1983). It can also be specified by the outer scale and the velocity variance, the second being also related to the energy levels involved. Both are theoretically related through the molecular viscosity of the fluid. We prefer the velocity variance for MST radar work, since it involves a radar measurable quantity, as compared to a theoretically derived ε, which involves certain assumptions.

For further reading see Batchelor (1953), Tennekes and Lumley(1972), Bolgiano(1968) and Tatarsky

Relationship between radar signals and atmospheric medium statistics

Signal statistics

The usefulness of a MST radar is based on the close relationship there is between the statistics of the signal received and the statistical properties of the atmosphere. It is our intention to show and discuss this relationship, its implications and limitations. Before we get into this task, let us first review the statistical nature of the signals received and ways to characterize their properties.

The experimental setup of an atmospheric radar has been covered by the previous lectures (See also Balsley and Gage,1980). Regardless of the possible variations of radar systems, it is convenient to think of the signals as a two dimensional process, but in which both dimensions have units of time. The idea is depicted in figure 3. The figure shows radar signal returns for a sequence of identical pulses. We are showing the signals after filtering and decoding, so we can still talk about identical pulses even if we have used a complementary pulse scheme. In one of the dimensions we have the delay time after the time of pulse transmission. On the other dimension, we have the time of pulse transmission. The process is discrete in this dimension. We can then represent the signal received as s(t,t'), where t stands for the (discrete) time at which the pulse was transmitted, and t' the delay time after the pulse. t' is continuous as an analogue output of the receiver, but in practice it is also discretized by the sampling and digital processing. As before we will be careless in differentiating the continuous vs. the discrete representation of signals.

It is convenient to make a change of variables and replace t' by 2h/c, where h stands for the radar range defined by the delay t', considering a pulse propagation at he speed of light, c. We can then write s(t,h) to describe the signal, dropping the 2/c factor from the notation for convenience. In this way we get around the disturbing dependence on two times as independent variables.

The radar signal is intrinsically a non-stationary time process as a consequence of the non-homogeneous nature of the atmosphere. By writing it in the form s(t,h) we have converted it into multiple (practically) stationary processes in time t, one for each range of interest. We can change our notation once more and write $s_h(t)$ to stress the parametric nature of h. We can now think of h as a label, labeling parallel processes, one for each altitude.

We know how to characterize a random stationary process: by its autocorrelation function. If the echoes come from a (practically) homogeneous turbulence, we can further argue using the limit theorem (sum of many independent contributors) that the process is Gaussian, in which case all the information we can extract from the process is in its autocorrelation function. Gaussian or not, $C_h(\tau)$ is defined as

$$C_h(\tau) = E[s_h(t)s_h(t+\tau)] \tag{14}$$

A good estimator of C_h is $\langle s_h(t)s_h(t+\tau)\rangle_n$ where the average has been evaluated by taking n pairs of sample points. Alternatively, as we have already seen, we can characterize the signals by its frequency power spectrum, $F_h(\omega)$, given by the Fourier transform of $C_h(\tau)$. Good estimates of $F_h(\omega)$ can be obtained from

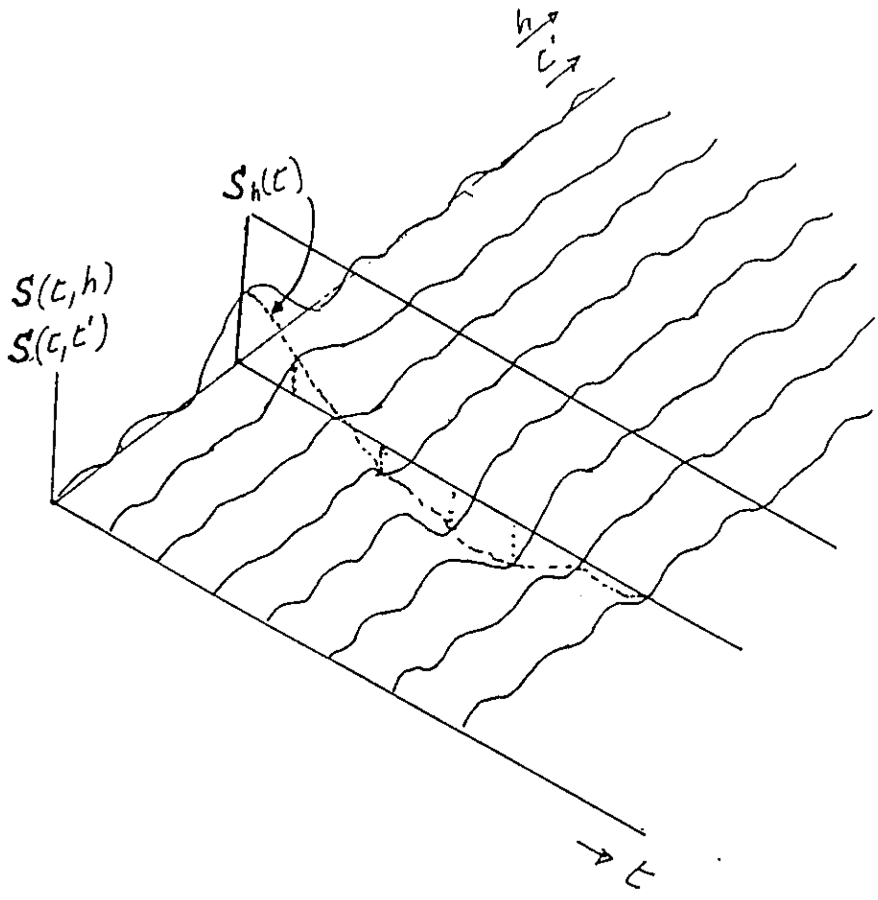

Figure 3 – Two dimensional schematic representation of the radar signals. t is the time of each radar pulse and t" the radar range delay. The process of interest is $S_h(t)$, i.e. the sampled signal at a given range, h, as a function of the time t of pulse transmission.

discrete Fourier transforms of $C_h(\tau)$ or directly from the sequence by the techniques that will be described later in the lectures.

So far we have considered in the introduction and the discussions above that the radar signals received are real. Indeed they are. We live in a real world. On the other hand, for practical reasons, the signals which originally have a frequency almost equal to the transmitter frequency are converted to base band. To preserve all of the information contained in the original signal we need two converted signals, The so call Q and I components (see lecture on radar hardware). It can be shown (e.g. Woodman and Kohl, 1976) that if we form a complex signal with the Q and I component as real and imaginary component, everything we have say is valid, if we replace $s(t)s(t+\tau)$ by $s(t)s^*(t+\tau)$. We can recover the statistics of the signals before baseband conversion by multiplying the correlation function by $\exp(j\omega_0 t)$, where ω_0 is the transmitter frequency, and then taking the real part. Any complex phase can then be interpreted as a real phase with respect to the transmitter frequency. In particular a Doppler shift in the received signal is manifested as a complex phase of the form $\omega_D t$ in the converted signal, and as a complex phase of the form $\omega_D \tau$ in the correlation function.

In the frequency domain, that is in the corresponding frequency power spectra, the effects are simpler , a spectrum of the form $F(\omega-\omega_0)$ is converted to a spectrum of the form $F(\omega)$. A Doppler shift shows as a displacement in both.

A general relationship

In the appendix we have derived a very general relationship between the statistical of a radar signal and the statistics of the fluctuations in density (we could have used the dielectric properties, the temperature, electron density or any other relevant linearly related quantity) of a scattering medium. There, we take the approach of considering the most general conditions the least amount of approximations. Particular cases allow further approximations and specific expressions that one can use in practice to estimate medium parameters or to discuss instrumental effects. It has the advantage of going from the most general to the particular keeping good track of the approximations involved and their limiting implications. Furthermore, it does not take any additional conceptual effort to derive the most general expression, namely

$$C(\tau, t) = \int d3\mathbf{x}d^3\mathbf{r}\ dt'd\tau'\ \chi(t;t',\mathbf{x})\ \chi^*(t+\tau;t'+\tau',\mathbf{x}+\mathbf{r})\ \rho(\mathbf{r},\tau';\mathbf{x}) \tag{15}$$

Cumbersome as it looks, because of the variety of arguments, the expression represents linear operations involving only two functions of easy interpretation, χ and ρ. ρ is the space– time autocorrelation function of the fluctuations responsible for the scattering. It characterizes the medium and depends only on the properties and dynamics of the medium. The function $\chi(t;\ t',\mathbf{x})$ may be called the "instrument function". It can be interpreted as the output of the instrument as a function of time as a consequence of a given arbitrary transmitter output shape (pulsed or continuous) having placed a point scatterer at point **x** in space, for an instant, at time t'. It is analogous to the impulse response of a system, although here the impulse is in the system characteristics: the scattering density.

The instrument function , χ, includes the pulse shape of the transmitter, any (amplitude, phase or frequency modulation) coding and decoding, match filtering , the geometry of the experiment, the transmitting and receiving characteristics of the antennas and the propagation properties of the medium, including any refraction if necessary. The determination of ρ is a statistical problem related to the physics of the medium. the determination of χ is an electronics and electromagnetics problem. As far as the characteristic of the medium, it includes non homogeneous and anisotropic cases. It is also valid for ionospheric radars including the incoherent scatter technique.

Although not discussed here or in the appendix, the approach can be extended easily to the case the system has two outputs, like in the case of a radar interferometer. We just replace the product of identical χ's by the product $\chi_a\chi_b^*$ where the a and b label stand for the outputs of the two antennas, or the two frequencies in a frequency domain interferometer (Kudeki and Stits , 1987)

At the appendix we have derived expressions which include explicitly the transmitter pulse shape , the receiver filter and decoding impulse response, and the antenna pattern. In order to perform some of the integrations and make discussion possible, we have also assumed that the scattering volume, defined by the antenna patterns and the effective pulse width is larger than the characteristic sizes of the fluctuations, although this assumption can be relaxed if necessary. It is possible to reduce the complexity of the expressions further, taking approximations which are valid for specific cases.

The MST case

In the case of MST or clear–air radars it is well justified to assume that the characteristic time of the medium is much larger than that of the pulse and matching filter. In that case we can use equation A.15 and write (with a slight change in notation):

$$C(\tau,h) = \int d^2 s dh' \; K^2(\mathbf{s},h') \; \phi(\mathbf{k}_s(\mathbf{s}),\tau;\mathbf{s},h') \; p(h-h')p^*(h-h'+\tau) \qquad (16)$$

It differs from the appendix notation in the use of h' for the range (delay) variable of integration and h for the "range" sampling time. We have also conveniently selected length units such that c/2 (half the speed of light) is unity. This allows us to use h and h' for a spatial as well as a time variable. The coordinate system of integration is defined by surfaces of equal delay and an arbitrary two dimensional coordinate, **s**, in the transverse direction. **k** is in the direction of h. The directional dependance of ϕ on **s** is shown explicitly. This dependance is important in the case of anisotropic turbulence and will be responsible for aspect sensitive effects. The possibility of non–homogeneous turbulence is also shown explicitly in the dependance of ϕ on h and **s**. This is important since it is known that turbulence occurs in layers thinner than the usual range resolution of the radar. The formula is valid for mono–static and bi–static radars, and K(**s**,h) stand for the product of the transmitter and receiver antenna weighing patterns. The dependance of K in h is usually slow (mainly the inverse of range squared) and can be taken out of the integral.

It is important to stress the fact that $\mathbf{k}_s$ is not the variable vector **k**; it is a constant vector defined by the vector difference of the incident and the scattered wave number vectors which characterize the incident and scattered electromagnetic wave which leaves the transmitter and arrives to the receiving antenna, respectively. In the case of a backscatter radar it has a wave number twice the corresponding wave number of the illuminating wave, and the same direction.

If we crosscorrelate, as we should, only samples which correspond to the same range, then we have an expression for the auto correlation of the time stationary process $s_h(t)$ we defined above. This is equivalent to restricting the time of the second sample to be at even multiples of the pulse repetition period. In which case, since the filtered pulse function p(t) is periodic, i.e. since p(t)=

p(t+nT), we can replace the product of displaced p's above by $|p(h-h')|^2$.

If we further assume that the medium is homogeneous in the transverse direction **s**, we get a simpler but yet very general expression for $C_h(\tau)$:

$$C_h(\tau) = \int d^2\mathbf{s}\ K^2(\mathbf{s}) \int dh'\ \phi(\mathbf{k}_s(\mathbf{s}),\tau;\ h')\ |p(h-h')|^2 \tag{17}$$

Before we continue with the discussion of this equation it is convenient to make one further approximation, discuss the results and then come back to this more general expression.

If we further assume that we have a homogeneous atmosphere in all directions, and that the antenna has a beamwidth much narrower than the characteristic angular width of any aspect sensitivity which $\phi(\mathbf{k}_s(\mathbf{s}),\tau)$ may present, we can take ϕ out of the integral and write

$$C(\tau) = B\ \phi(\mathbf{k}_s,\tau) \tag{18}$$

or

$$F(\omega) = B\ \Phi(\mathbf{k}_s,\omega). \tag{19}$$

The success of radars to study the atmosphere is based on these simple formulae. Even in the case that the approximations behind them are not quite valid, its discussion allows us a first order approximation of the results. We shall discuss the significance of this equation first, and then remove some of the approximations that make it valid.

We will discuss only the implications of the terms ϕ or Φ on above equations. Since both expressions are interrelated, we will most of the time limit our discussions to the time domain expression, i.e equation (18) and extend it to the frequency domain (equation (19)) when desirable. We will not discuss the proportionality term, B, since that is equivalent to a discussion of the radar equation, which we have already seen in the previous lectures.

The first conclusion we can derive from these expressions is that the amplitude and dynamics of the radar signal depends linearly on the amplitude and dynamics of only one Fourier component of the density fluctuations of the medium, that which has a wave-vector equal in amplitude and direction to twice (backscatter case)the wave-vector of the probing electromagnetic wave. In terms of wavelengths, the radar is sensitive only to fluctuations with a wave length half the wavelength of the probing wave and a direction equal to the line of sight. The radar effectively filters out very sharply all the spatial Fourier components which are not equal to $\mathbf{k}_s$. This wave component is still a random process. Its dynamics is characterized by its temporal correlation function, $\phi(\mathbf{k}_s,\tau)$. The signal received has the same dynamics as this particular wave component.

The "power" , i.e. the amplitude squared averaged, of the particular wave component of the density fluctuations the radar is sensitive to, is given by $\phi(\mathbf{k}_s,0)$. Therefore the power of the radar signal is proportional to the "power" of the same spatial wave component. Furthermore, if we assume that the k–spectrum follows a Kolmogorov law, we can indirectly infer the power density at other wavelengths.

If the medium is inmoble with respect to a frame of reference, in this frame of reference we can show that $\phi(\mathbf{k},\tau)$ is real. This is a consequence of the invariance of $\rho(\mathbf{r}, \tau)$ under an interchange of $\mathbf{r}$ with $-r$ for any τ. If it were not invariant we would violate our inmoble assumption since there would be dynamically a preferred direction. An observer moving with respect to this reference at velocity $\mathbf{v}$ would measure instead a correlation function of the form $\rho(\mathbf{r}\text{-}\mathbf{v}\tau, \tau)$, as a consequence of a transformation $\mathbf{x}' = \mathbf{x}-\mathbf{v}t$ in the defining equation (6) for ρ. Using the displacement theorem of Fourier transform pairs, we derive a $\mathbf{k}$–spectrum of the form $\phi(\mathbf{k}, \tau)\exp(-j\mathbf{k}_s.\mathbf{v}\tau)$. Replacing this spectral form in equation (18), and remembering that $\Phi(\mathbf{k}_s, \tau)$ is real,we conclude that the phase slope of the signal correlation is a measure of the projected velocity of the medium with respect to the radar. The projection is along $\mathbf{k}_s$. In terms of the frequency power spectrum $F_h(\omega)$, again using the displacement theorem, we get a new expression, $F_h(\omega-\omega_D)$, where ω_D is , not surprising, the Doppler frequency,

$$\omega_D = \mathbf{k}_s.\mathbf{v} = (v_k/2c)\omega_o \ . \tag{20}$$

Our next step is to show that the characteristic time of the signal correlation is determined by the variance, $\langle w'^2\rangle$, of the turbulent velocity. This is better shown in the frequency domain. If the scattering volume is larger than the largest eddies, we are sure to have a good sample of all possible velocities within the volume. Normally the eddies are much larger than the wavelength of the fluctuations the radar is sensitive to. We can then divide the scattering volume into many scattering sub–volumes. The signal received would be equal to the sum of each of the contributions of these sub–volumes, each of which would impose a Doppler shift proportional to its averaged projected velocity w" . This projected velocity would not differ much from a corresponding local w', since we know from turbulence theory that most of the energy is in the largest scale eddies. Therefore the power frequency distribution (spectrum) of the backscattered signals is going to be distributed in the same way as the probability distribution of w' . Its second moment, σ^2, would be proportional to the variance of the velocity $<w'^2>$, with the same constant of proportionality as the one which relates the velocity to the Doppler shift, but squared, namely

$$\sigma^2 = \omega_o^2<w'^2>/4c^2 \ . \tag{21}$$

Furthermore, we have mentioned before that from experimental results as well as from limit theorem arguments, we expect the random turbulent velocities to be normally distributed, therefore, we also expect the frequency power spectrum of the radar signals to be distributed likewise.

A normal frequency power spectrum is defined by three parameters: its area(total power), displacement and width; or, alternatively, by its three first moments. It transforms to an autocorrelation function which is also normal, although complex. The three parameters transform into : the amplitude , phase slope and width of the autocorrelation function, respectively. That is all the statistical information either one of them contains, and that is all we should look for in this case. On the other hand we have seen that they are related to very important parameters of the medium. In fact, the relation and importance holds even if normality is not assumed.

Let us come back to the more general equation, (17). The whole expression can be taken as a weighted averages of ϕ, averaged over all ranges weighted by the filtered pulse shape squared, and over all angles weighted by the antenna pattern. In the case of a bi-static arrangement, the averages are taken over surfaces of equal delay ("range") and over appropriate transverse coordinates ("angle").

The pulse function is non-zero for values close to h–h'=0 , and a depth equal to its width after convolving it with the filter function (similar shape for matched conditions). this means that the range integral is effectively sampling Φ at h'=h , averaging neighboring values within approximately a pulse width.

Similarly, the antenna weighing function is non-zero for values close to the axis of the beams, and a width given by the beamwidth of the antennas.

If the dependance of ϕ on **s** or h is relatively slow as compared to the width of the weighing functions p^2 and K^2, an average value of ϕ, representative of the center point of the sampled scattering volume at range h and center angle of the beam patterns $\mathbf{s}=\mathbf{s}_0$, can be taken out of the integral. The integrand is reduced to the two weighing functions, which integrate to a volume V, as large as the non zero regions of p^2 and K^2, multiplied by the proportionality constants imbedded in them. The result,

$$C_h(\tau) = b\ V\ \overline{\phi}(\mathbf{k}_s,\ \tau;\ h), \qquad (22)$$

is a proportional expresion as the one in (18) and (19), which we have already discussed. We have replaced $\mathbf{k}_s(\mathbf{s}_0)$ by $\mathbf{k}_s$, where k_s stands for the corresponding one at the center of the beam. The only difference being the explicit linear dependance on the volume, V, and the averaging nature of the integral operation

An important use of equation (17) is in the evaluation and discussion of broadening of the spectrum, $F(\omega)$, as a consequence of finite beamwidth and wind shear. The evaluation should be done by actually using the equation, and a model of the medium characteristics and the radar system in the integrand. But, it is possible to get a feeling of how the broadening comes about by breaking the integral into the sum of integrals over smaller volumes sufficiently small for equation (18) to be valid. Each subvolume will contribute to the spectrum with comparably shaped spectra but with different Doppler shift, $\mathbf{k}_s.\mathbf{v}$. The Doppler shifts would be different either because $\mathbf{k}_s$ varies in direction within the beamwidth (beam broadening) or because $\mathbf{v}$ varies (shear broadening). The resultant spectrum would be significally wider if these shifts are larger than the ones produced by the random turbulent velocities. (See Hocking, 1983, for further discussions).

Notice here that it is possible for non isotropic turbulence to have a Φ dependent on $\mathbf{s}$ through its dependance on the direction of $\mathbf{k}_s$, that is an aspect sensitivity. If the aspect sensitivity is wider than the beamwidth, the radar would be able to resolve it and actually measure the angular dependance, provided of course that the beam is steerable. If the aspect sensitivity is sharper than the beam pattern, then the weighing in the integrand will be performed by the aspect sensitivity function , and the statistics of the echoes will be mainly that corresponding to the most favored aspect angle. The contributing volume will also be correspondingly smaller.(See Doviak and Zmic, 1984, for further discussions).

Something similar would happen if the h dependance of ϕ is smaller than the pulse width. The most important consequence being that the volume would be smaller than that defined by the pulse. Thus, the actual strength of turbulence, $\phi(k)$, would be underestimated if the h dependance of ϕ is not taken into account.

Partial reflection.

So far we have considered only radar echoes that have been produced by random turbulent–produced fluctuations in refractive index. It is possible to have in the atmosphere stratified structure sufficiently large in the horizontal extent as to be considered deterministic for all practical purposes. In fact, the aspect sensitivity that has been measured is so sharp that has let some researchers (Röttger and Liu,1978 ; Fukao et al,1979; Gage and Green 1978) to postulate that the echoes are produced by partial reflection from stratified gradients. In this case is more convenient to talk, borrowing from optics, about the reflectivity of the structure, R. It is a coefficient, defined by the ratio of the intensity of the reflected over the incident electromagnetic wave, incident on the structure. A formula often used in the literature to evaluate R is

$$R = \int_{-L/2}^{L/2} \frac{1}{n} \frac{dn}{dz} \exp(-jk_s z)\, dz \, .$$

Recently, Woodman and Chu, 1988, have shown that the limits, L/2, if they fall at points where the integrand has not gone to zero on its own, can introduce artificial discontinuities in the first derivative which overestimate the reflectivity by many orders of magnitude. Nevertheless, partial reflection is possible if step like structure of a fraction of a degree Kelvin exist within a length scale of a meter or so. The existence or not of such a discontinuous structure would have to be established with an independent technique. The aspect sensitivity observed with radars can also be explained in terms of anisotropic turbulence at the edges of the turbulent layers observed with the same technique (Woodman and Chu, 1988).

Characteristics of noise and clutter interference

Radar echo signals are always contaminated, in variable degrees, with sky and receiver noise and echoes from undesirable targets, like mountains, other ground structures, ocean waves, etc.. The latter is referred as clutter. In order to

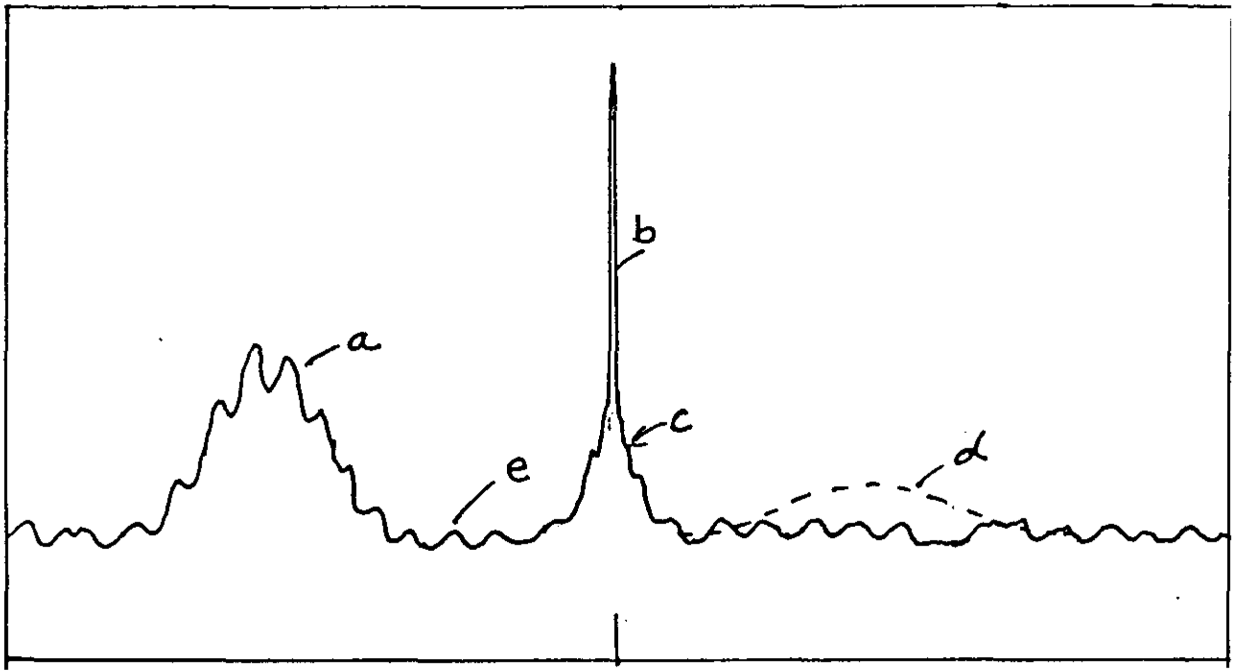

Figure 4 – Schematic plot of a typical VHF MST radar signal showing a) the signal component, b) non–fading ground clutter, c) fading ground clutter, d) possible ocean clutter and e) receiver and sky noise.

properly interpret the desired signals, and be able to discriminate between them and noise or clutter, we need to know the spectral characteristics of the latter as well.

Sky and receiver noise, after passing through the receiver, has a bandwidth determined by the receiver filter. The filter in turn is normally matched to the transmitter pulse width, or Baud width if coded. The pulse width is a small fraction of the pulse repetition period, which also determines the sample time of the sequence $s_n(t)$. Therefore, at this sampling rate, the noise samples are independent. They are also statistically independent with respect to the signal. Hence, the noise contribution to the autocorrelation function of the received signals is a Dirac function centered at the origin. Its contribution to the frequency power spectrum is a flat threshold. It behaves, then as white noise.

The characteristics of ground clutter are the opposite to those of noise. They are very narrow in the frequency domain and wide in the time domain. To first approximation clutter shows as an spectral line in the frequency domain, centered at zero frequency, since it comes from rigid structures with no relative velocity with respect to the radar. At low VHF frequencies, this is practically the case. At UHF frequencies, the reported clutter characteristics (Sato and Woodman, 1981) have two components, an spectral component accompanied by a weaker narrow, but finite, width component both centered at zero frequency. The spectral line comes as in the VHF case from the rigid ground structures, the wider component is believed to come from wind induced motion of tree branches or from phase modulation of the spectral component induced by changes in the effective phase path length between the radar and the target. Both are possible. Changes in the width of this component with different surface wind conditions support them both. Fortunately, except under very windy conditions, the wider component is still a few to several times narrower that the width of the atmospheric echoes and one can discriminate against them (Sato and Woodman, 1981). The task is made easier by its confinement to the center of the spectrum. Under windy conditions, specially when one is interested in the small vertical component, ground clutter is a problem at UHF frequencies.

For those radars near the ocean or large lakes, ocean clutter is a source of interference. It can compete in strength with the atmospheric echoes, specially at the higher ranges. Ocean clutter comes from wavelets on the surface with a wave length equal to half the wavelength of radar. It is Doppler shifted by a frequency corresponding to the phase velocity of the wavelet. This velocities are of the order of a few meters per second, and hence comparable to the atmospheric velocities we are interested in. This should not surprise us, since the wavelets are exited by matching velocity components the surface wind speed. To make matters worse, ocean clutter echoes have spectral widths which are also comparable to that of the desired echoes (Sato and Woodman, 1982b) . Still it is possible to discriminate against them, taking advantage of the predictable frequency shift and their constancy – in amplitude and frequency– as a function of range and time. The problem being limited to those altitudes where the wind profile crosses the value corresponding to the velocity of the wavelets, and only in the case its strength is comparable or weaker to the interference.

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APPENDIX

Scattering of EM Waves from Dielectric Density Fluctuations*

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Radars are used for remote probing of the upper atmosphere. Monostatic and bistatic configurations have been used. The echoes are obtained from the scattering of the illuminating wave by fluctuations in the dielectric properties of the medium under study.

The fluctuations in the local dielectric constant of a medium are direct consequences of fluctuations in the density of the medium or, more properly, on the density of that component or components in the medium responsible for its dielectric behavior, e.g., electron density in an ionized gas, "air" density and water vapor in the low atmosphere, etc.

In the case the medium is in thermodynamic equilibrium, the fluctuations are reduced to a minimum (thermal level). In such a case, and for an ionized plasma, we refer to the technique as incoherent scatter. These fluctuations are never at zero level due to the discrete nature of matter (Summations of delta functions will always produce fluctuations.)

Density fluctuations are statistically characterized by the density space-time correlation function $\rho(r, \tau, \underset{\sim}{x})$ defined as

$$\rho(\underset{\sim}{r},\tau;\ \underset{\sim}{x}) \equiv \langle n(\underset{\sim}{x},t)\, n(\underset{\sim}{x} + \underset{\sim}{r}, t + \tau) \rangle \tag{1}$$

where n(x,t) is the microscopic random density of the medium at position x in space and time t. In (spatially) homogeneous medium ρ is independent of x and $\rho(r,\tau) \equiv \rho(r,\tau;\ x)$.

Hagfors has treated the problem of how to find $\tilde{\rho}(x,\tau)$ for an ionized medium in thermodynamic equilibrium (or quasi-thermodynamic for the case $T_e \neq T_i$). Farley has described the different techniques for obtaining estimates of $\rho(r, \tau;\ x)$ from the scatter echoes.

We shall develop here the functional relationship that exists between the statistical characterization of the signal received in a radar experiment and the fluctuations in the medium characterized by $\rho(r, \tau;\ x)$. The fluctuations need not be at the thermal level, so we are not limited to the incoherent scatter problem. We should point out that the usefulness of large radars for the study of the upper atmosphere is not limited to incoherent scatter. Proof of which is found in the large number of papers produced by the Jicamarca Observatory by studying backscatter echoes from E- and F-region irregularities and from turbulent fluctuations in the neutral atmosphere. In fact, some smaller radars are built (STARE, SOUSY and the TS radars) which depend on the enhanced reflectivity produced either by instabilities or turbulence. This could be the case in EISCAT when observing auroral phenomena or the effects of artificial heating. It will also be the case when studying neutral dynamics using backscatter signals from turbulent fluctuations.

Said functional relationships can be found in the literature but it is usually derived from very simplified conditions with assumptions which are not necessarily valid. The derivation is usually heuristic and in many cases difficult to assess the range of validity of the derived expressions. Such approach is, of course, useful for didactic purposes and when the purpose of the paper is on other aspects of the problem. Derived expressions in the literature are usually derived for a specific technique (out of the many described here by Farley) and for specific conditions (e.g., homogeneous media, continuous illumination, slowly varying echoes, narrow pulses, etc.). We shall derive here the functional relationship between the statistical properties of the echoes and the statistical properties of the medium under very general conditions.

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*Lecture presented at the M.P.I. EISCAT School, January 1979, Oberstdorf, W. Germany

We shall consider an experimental configuration as depicted in Figure 1. The medium under study is illuminated by an EM wave of frequency w_0, modulated by an arbitrary complex signal p(t), scattered EM waves are received at a different location (or at same as a particular case), coherently detected, properly filtered and decoded (if necessary). We are interested in evaluating the complex autocorrelation of the signal received, O(t), i.e.,

$$C(\tau,t) \equiv \langle O(t)\, O^*(t+\tau) \rangle \tag{2}$$

in terms of the space and time density correlation of the medium.

The signal O(t) is a random process, usually nonstationary, is fully characterized by its time autocorrelation function $C(\tau,t)$. The dependence on t can normally be associated with a given range, h, corresponding to the delay.

We assume: (1) that there is only primary scattering (first Born approximation valid), i.e., the medium is transparent, the illuminating field at a point x within the medium is due to the primary illuminating field and the scattered fields at x are negligible; (2) the system is linear, i.e., if $O_1(t)$ is received for $\rho_1(t)$ and $O_2(t)$ for $\rho_2(t)$. The $\alpha O_1(t) + \beta O_2(t)$ is received for an excitation $\alpha p_1(t) + 5\, P_2(t)$. The linearity of the propagation in the medium is guaranteed by the linearity of Maxwell equations.

The linearity of the system allows us to evaluate the output signal as the linear superposition of the contributions of each differential volume, $d^3\underset{\sim}{x}$ with density $n(x,t)$. This differential contribution can be evaluated in terms of the linear operators depicted in Figure 2. Here we have modeled the propagation of the transmitter to the scattering point by a delay operator with delay $T_t(\underset{\sim}{x})$ and an amplitude factor $K_1(x)$ which represent the effect of antenna gain and other system parameters. The scattered signal is proportional to the local instantaneous (random) density $n(x,t)$ of the medium times the volume d^3x. The dielectric properties of the medium, the receiver, antenna, and other propagation properties are contained in a constant gain (in time) $K_2(x)$. There is a delay block with delay $T_2(x)$, a detector and a filter before we finally get our output from the differential contribution from $n(x,t)$. The filter is characterized by the complex input response h(t) and includes any decoding scheme. Decoding is a convolution operation and can be considered as part of the filter.

The evaluation of the delay functions $T_1(x)$, $T_2(x)$ and the constant terms $K_1(x)$, $K_2(x)$ does not concern us here and are assumed to be known. The output of the system can then be written as

$$o(t,\underset{\sim}{x})d^3\underset{\sim}{x} = d^3\underset{\sim}{x} \int dt'\, K(\underset{\sim}{x})\, p(t' - T(\underset{\sim}{x}))\, e^{-i\, w_0 T(x)}\, n(\underset{\sim}{x}, t' - T_2(\underset{\sim}{x}))h(t - t') \tag{3}$$

where we have already operated on the "signal" with the delay operators $\delta(t - T_1(x)$ and $\delta(t - T_2(x))$. Here we have used $T(x) = T_t(\underset{\sim}{x}) + T_2(\underset{\sim}{x})$ for the total delay and $K(\underset{\sim}{x}) = K_t(\underset{\sim}{x}) \bullet K_2(\underset{\sim}{x})$. The total signal output is then

$$O(t) = \int d^3\underset{\sim}{x}\; o(t,x) \tag{4}$$

and the autocorrelation, $C(\tau,t) \equiv \langle O(t)\, O^*(t+\tau) \rangle$, can then be written as:

$$C(\tau,t) = \int d^3\underset{\sim}{x}\, d^3\underset{\sim}{x}' dt' dt'' K(\underset{\sim}{x})\, K(x')\, p(t' - T(\underset{\sim}{x}))\, p^*(t'' - T(\underset{\sim}{x}) e^{-i\, w_0 (T(\underset{\sim}{x}) - T(\underset{\sim}{x}'))}$$

$$\bullet\, h(t - t')\, h(t + \tau - t'')\, \rho[\underset{\sim}{x}' - x,\, t'' - t' - (T_2(\underset{\sim}{x}) - T_2(\underset{\sim}{x}')) ;\, \underset{\sim}{x}] \tag{5}$$

It is convenient to write this expression in terms of variables

$$\underset{\sim}{r} = \underset{\sim}{x}' - \underset{\sim}{x}$$

$$\tau' = t'' - t'$$

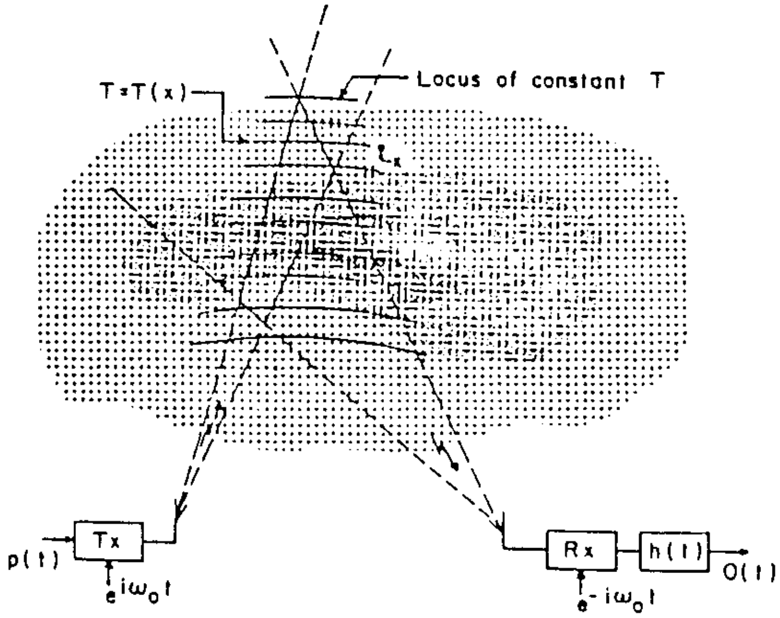


Figure 1.

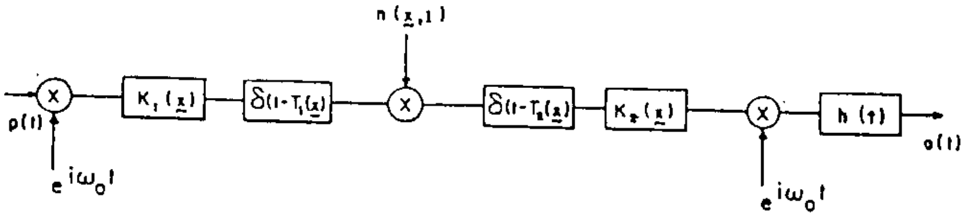


Figure 2.

$$C(\tau,t) = \int d^3\underset{\sim}{x}\, d^3\underset{\sim}{r}\, dt'\, d\tau'\, K(\underset{\sim}{x})\, K^*(\underset{\sim}{x}+\underset{\sim}{r})\, p(t'-T(\underset{\sim}{x}))\, p^*(t'+\tau'-T(\underset{\sim}{x}+\underset{\sim}{r}))$$
$$\cdot\, e^{-iw_o(T(\underset{\sim}{x})-T(\underset{\sim}{x}+\underset{\sim}{r}))}\, h(t-t')\, h^*(t+\tau-t'-\tau')\, \rho[\underset{\sim}{r},\tau'-(T_2(\underset{\sim}{x})-T_2(\underset{\sim}{x}+\underset{\sim}{r}));\underset{\sim}{x}] \quad (6)$$

This expression is simplified considerably if we take advantage of the fact that in most cases the characteristic length of the density correlation function, r_c, (equal to the Debye length in the I.S. case) is much smaller than the characteristic length of $K(\underset{\sim}{x})$ and the characteristic length, $c\tau_p$, corresponding to the width of the pulse $p(t)$. This allows us to replace $K(\underset{\sim}{x}+\underset{\sim}{r})$ by $K(\underset{\sim}{x})$ and $p(t - T(\underset{\sim}{x}+\underset{\sim}{r}))$ by $p(t-T(\underset{\sim}{x}))$ in the integrand with no appreciable effect on the integral.

Also, the difference in propagation time $T_2(x) - T_2(\underset{\sim}{x}+\underset{\sim}{r})$ is of the order of r_c/c for points within a correlated volume. This is much smaller than the characteristic time of the decay of the correlation function unless one is dealing with relativistic plasma. Therefore we can ignore this term in the time argument of the correlation function. In addition, the oscillatory nature of the exponential, with a wavelength comparable to the wavelength of the probing wave, makes the integrand unsensitive to any possible long scale structure of the correlation function across the surfaces of constant T.

Furthermore, the almost linear behavior of $T(\underset{\sim}{x}+\underset{\sim}{r})$ on $\underset{\sim}{r}$ for $|\underset{\sim}{r}| < r_c$ allows us to linearly expand $T(\underset{\sim}{x}+\underset{\sim}{r})$ in the exponent around $\underset{\sim}{x}$ and write:

$$w_o T(\underset{\sim}{x}+\underset{\sim}{r}) = w_o\, T(\underset{\sim}{x}) + w_o \nabla_{\underset{\sim}{r}}\, T(\underset{\sim}{x}) \cdot \underset{\sim}{r} = w_o\, T(\underset{\sim}{x}) + k(\underset{\sim}{x}) \cdot \underset{\sim}{r} \quad (7)$$

where $\underset{\sim}{k}(x) = \underset{\sim}{k}_1(x) - \underset{\sim}{k}_2(x)$, and $\underset{\sim}{k}_1(\underset{\sim}{x})$ and $\underset{\sim}{k}_2(\underset{\sim}{x})$ are the local wave number of the incident and scattered wave, respectively. With this approximation we can write:

$$C(\tau,t) = \int d^3x\, dt' d\tau'\, K^2(\underset{\sim}{x})\, p(t'-T(\underset{\sim}{x}))\, p^*(t'+\tau'-T(\underset{\sim}{x}))$$
$$\cdot\, h(t-t')\, h^*(t+\tau-t'-\tau')\, \hat{\rho}(\underset{\sim}{k}(\underset{\sim}{x}),\tau';\underset{\sim}{x}) \quad (8)$$

where $\rho = (\underset{\sim}{\xi},\tau;\underset{\sim}{x})$ is the spatial Fourier transform of $\rho(\underset{\sim}{r},\tau;\underset{\sim}{x})$ defined by

$$\rho(\underset{\sim}{\xi},\tau;\underset{\sim}{x}) = \int d^3\underset{\sim}{r}\, e^{-i\underset{\sim}{\xi}\cdot\underset{\sim}{r}}\, (\underset{\sim}{r},\tau;\underset{\sim}{x}) \quad (9)$$

Notice that as far as $\underset{\sim}{r}$ is concerned, $\underset{\sim}{x}$ can be considered as a constant parameter. Also notice that the output of the experiment depends only on the Fourier component evaluated at a particular set of wave numbers $\underset{\sim}{k}(x)$, which for most cases is a constant. It is equal to $2k_1$ in the backscatter case. Equation (8) is the general expression we are after; it involves only two basic assumptions and one approximation regarding the length scale of $\rho(\underset{\sim}{r})$. It can be used as the starting point for simpler expressions applicable to the particular cases.

Next we consider a few particular cases as illustrative examples.

Case 1. Continuous excitation.

In the case of a cw bistatic experiment, e.g., the French incoherent scatter radar, we have $p(t) = a$, where a is a constant.

In such a case the output of the experiment is time stationary and the correlation function, $C(\tau) \equiv C(\tau,t)$, is given by

$$C(\tau) = a^2 \int d^3x\, K^2(\underset{\sim}{x}) \int d\tau'\, \hat{\rho}(\underset{\sim}{k}(\underset{\sim}{x}),\tau',\underset{\sim}{x})\, \phi_{hh}(\tau-\tau') \quad (10)$$

where the second integral is the usual convolution of the correlation function of the input signal to a filter by the autocorrelation function, $\phi_{hh}(\tau)$ of the filter characteristic. The spatial integral represents a weighted average of the contributions of each differential volume, weighted by the beam patterns of the antenna (and the $1/R^2$ dependence). For homogeneous media and constant $\underset{\sim}{k}(\underset{\sim}{x}) = \underset{\sim}{k}$, the spatial integral is independent of ρ and defines a volume, V, and we have

$$C(\tau) = a^2K^2\, V \int \hat{\rho}(\underset{\sim}{k},\tau)\, \phi_{hh}(\tau - \tau')\, d\tau' \tag{11}$$

The above equations, if expressed in the frequency domain, take an even simpler form where the convolution integral is transformed to a product of frequency functions.

Case 2. Filter time scale smaller than characteristic time of ρ.

In this case the integrand is different from zero for small values of the argument of $h(\cdot)$, i.e., when

$$t \approx t'$$

$$\tau \approx \tau' + t' - t$$

Thus, $\rho(\underset{\sim}{k}(\underset{\sim}{x}), \tau'; \underset{\sim}{x})$ can be taken out of the τ' integral evaluated at $\tau' = \tau$. We can then write (8) as

$$C(\tau,t) = \int d^3\underset{\sim}{x}\, K^2(\underset{\sim}{x})\, \hat{\rho}(\underset{\sim}{k}(\underset{\sim}{x}), \tau; \underset{\sim}{x})\ \bar{p}(t - T(\underset{\sim}{x}))\, \bar{p}^*(t + \tau - T(\underset{\sim}{x})) \tag{12}$$

where $\bar{p}$ is defined as

$$\bar{p}(t) = \int dt' p(t')\, h(t - t') \tag{13}$$

that is the pulse shape passed through the filter or decoder. In optimum designs h(t) is identical to p(t), and $\bar{p}(t)$ is then the autocorrelation of the pulse shape. In multiple pulse experiments the filter is identical to a pulse element of the sequence and $\bar{p}(t)$ is a sequence of autocorrelated pulses.

Surface of constant delay, $T = T(x)$, can be used as one of the variables of integration (e.g., range in a backscatter case with plane wave fronts) and a suitable set of two transverse coordinates, $\underset{\sim}{s}$, for the remaining two. We can then write:

$$d^3\underset{\sim}{x} = d^2\underset{\sim}{s}\, c dT \tag{14}$$

where c is the local phase velocity of light taken to be a constant for simplicity, $d^2\underset{\sim}{s}$ is a surface differential. Equation (2) then takes the form

$$C(\tau,t) = c \int d^2\underset{\sim}{s} \int dT\, K^2(\underset{\sim}{s},T)\, \hat{\rho}(\underset{\sim}{k}(\underset{\sim}{x}), \tau; \underset{\sim}{s},T)\, \bar{p}(t - T)\, \bar{p}^*(t - T + \tau) \tag{15}$$

Case 3. Backscattering from a (quasi-) homogeneous andisotropic medium.

This case illustrates the effect of decoding and filtering on the dependence of the autocorrelation function. The assumptions involved allow us to replace $\hat{\rho}(k(x), \tau'; x)$ by $\hat{\rho}(k, \tau')$ and to take it out of the spatial integral. For quasi-homogeneous cases we can take $\hat{\rho}(k, \tau; x)$ with the value it has at the center of the volume, which corresponds to the particular delay $\tilde{t}$ of the measurement. Therefore we will write $\rho_t(k,\tau')$ to extend the generality.

We can also perform the spatial integral in terms of the variables s and T. Only $K^2(x)$ is a function of $\underset{\sim}{s}$ and we can perform the integral with respect to this variable. If K^2 is a factor which groups all the dimensional factors in $K^2(x)$ then the spatial integral gives us $K^2A(T)$, where $A(T)$ is an equivalent area defined by the s dependence of the beam pattern. On most cases of interest $A(T)$ is a slowly varying function of T, slower than the pulse length and can be taken out of the integral evaluated at the sampling delay t. Considering the above we write equation (8) as

$$\begin{aligned} C(\tau,t) &= CK^2A(t)\int d\tau' dt' dT\, \hat{\rho}_t(k,\tau')\, p(t'-T)\, p\bullet(t'+\tau'-T)\, h(t-t')\, h(t+\tau-\tau'-t') \\ &= CK^2A(t)\int d\tau' \rho_t(k,\tau') \int dt'\, h(t-t')\, h^*(t-t'+\tau-\tau') \int dT\, p(t'-T)\, p^*(t'+\tau'-T) \end{aligned} \tag{16}$$

or

$$C(\tau,t) = CK^2A(t)\int d\tau' \; \hat{\rho}_t(k,\tau')\, \phi_{pp}(\tau')\, \phi_{hh}(\tau-\tau') \tag{17}$$

where $\phi_{pp}(\tau)$ is the autocorrelation function of the pulse shape and $\phi_{hh}(\tau)$ the autocorrelation function of the filter and decoding system.

Illustrative Examples

In order to gain a better understanding of the significance of the formulas derived for cases 2 and 3, we have constructed Figures 3 and 4, respectively, corresponding to two often used pulse schemes. Case 1 does not need an illustration since in this case the spectrum of the signal received is just the product of the spectrum of the medium with the systems filter characteristics.

Figure 3 depicts the different shapes of the functions involved for a double-pulse experiment, in a backscatter mode, in which two narrow pulses are sent, τ_s apart. In this case the experiment provides information on the correlation function $\hat{\rho}_t(k\ \tau')$, at only one delay, $\tau = \tau_s$, corresponding to the pulse separation. In practice the correlation function is evaluated only at this delay. To obtain the value of the correlation function at other delays, another pair of pulses is sent with the proper spacing.

Notice that $C(\tau,t)$ is different from zero only in the vicinity of τ_s, the useful part, and in the vicinity of $\tau = 0$ corresponding to a power measurement. Such power measurement is not useful since it contains not only the contribution from the desired height but also the "self-clutter" contribution from $t - \tau_s$, as illustrated in the two-dimensional plot of $\bar{p}(t-T)\, \bar{p}^*(t-T+\tau)$.

Multiple pulse schemes can be illustrated in a similar fashion, the main difference being that several correlation delays can be estimated in a single sequence and that the self-clutter is larger and coming from several different altitudes.

Figure 4 illustrates the case in which a long pulse (as compared to the medium correlation times) is sent. The receiver impulse response is narrow and considered square for the sake of simplicity. Two effects are clear from the picture, the medium correlation function is multiplied by a triangular function, $\phi_{pp}(\tau)$, and the result convolved with a narrower function, $\phi_{hh}(\tau)$, given by the self-convolution of the filter input response.

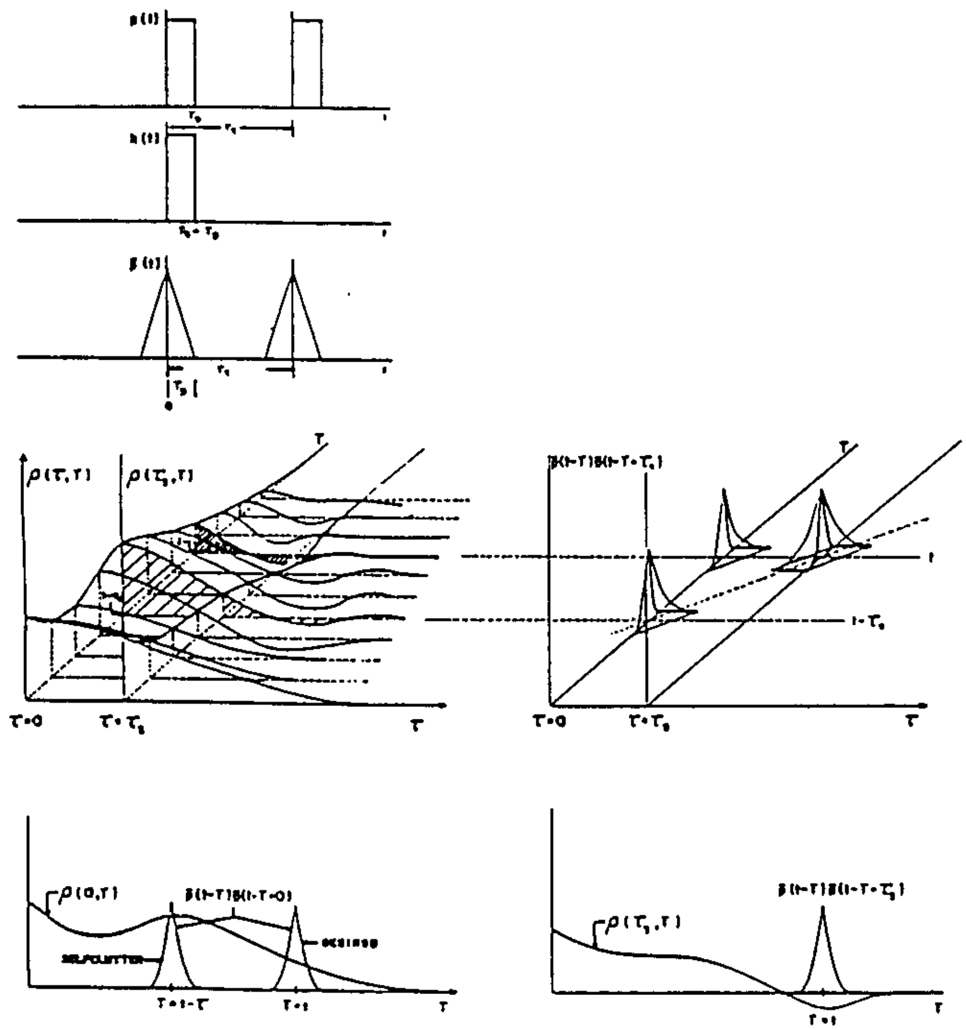
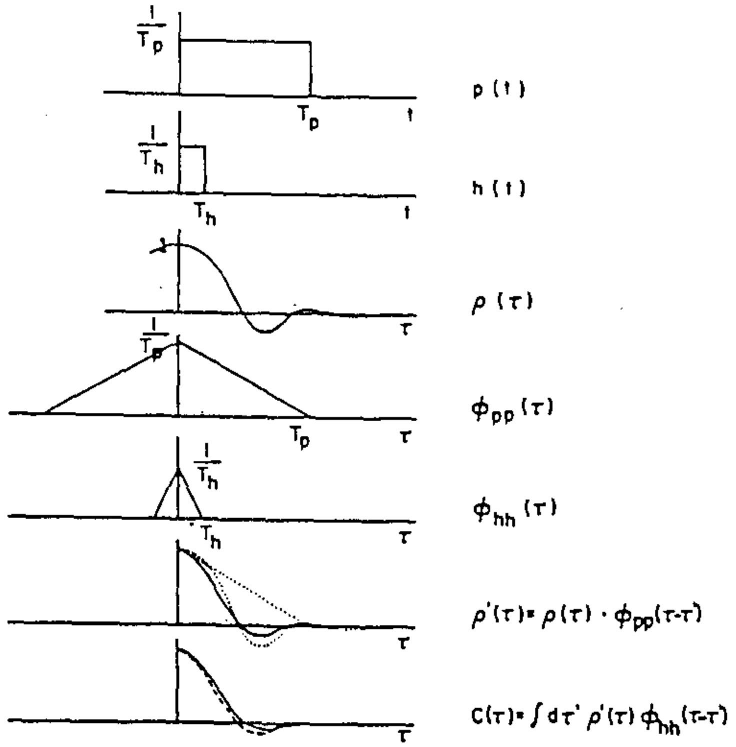


Figure 3.



$\frac{1}{T_p}$
T_p
t
p (t)
$\frac{1}{T_h}$
T_h
t
h (t)
τ
ρ (τ)
$\frac{1}{T_p}$
T_p
τ
$\phi_{pp}(\tau)$
$\frac{1}{T_h}$
T_h
τ
$\phi_{hh}(\tau)$
τ
$\rho'(\tau) = \rho(\tau) \cdot \phi_{pp}(\tau - \tau')$
τ
$C(\tau) = \int d\tau' \rho'(\tau) \phi_{hh}(\tau - \tau')$

Figure 4.