Inverse Methods in Aeronomy

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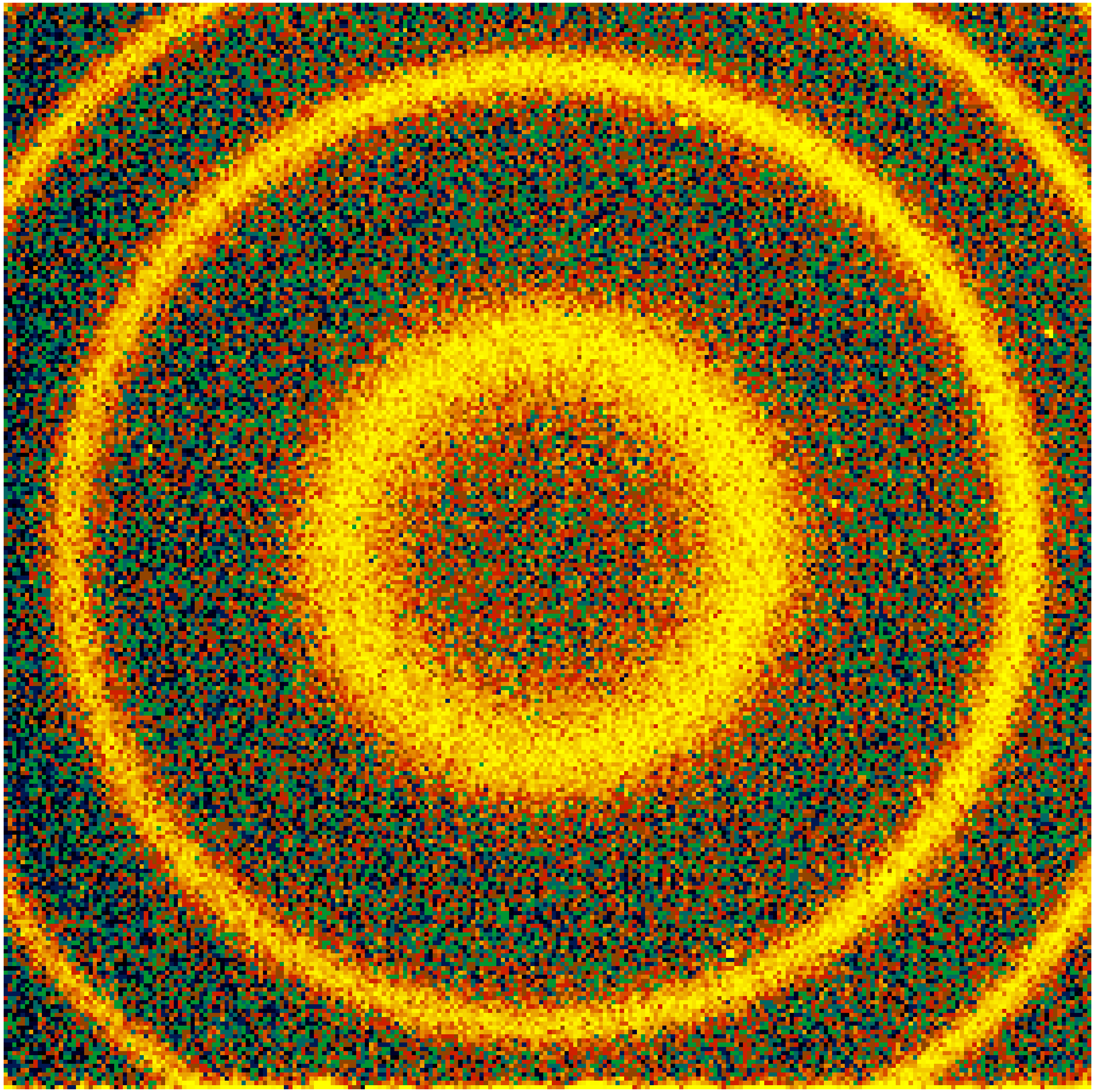
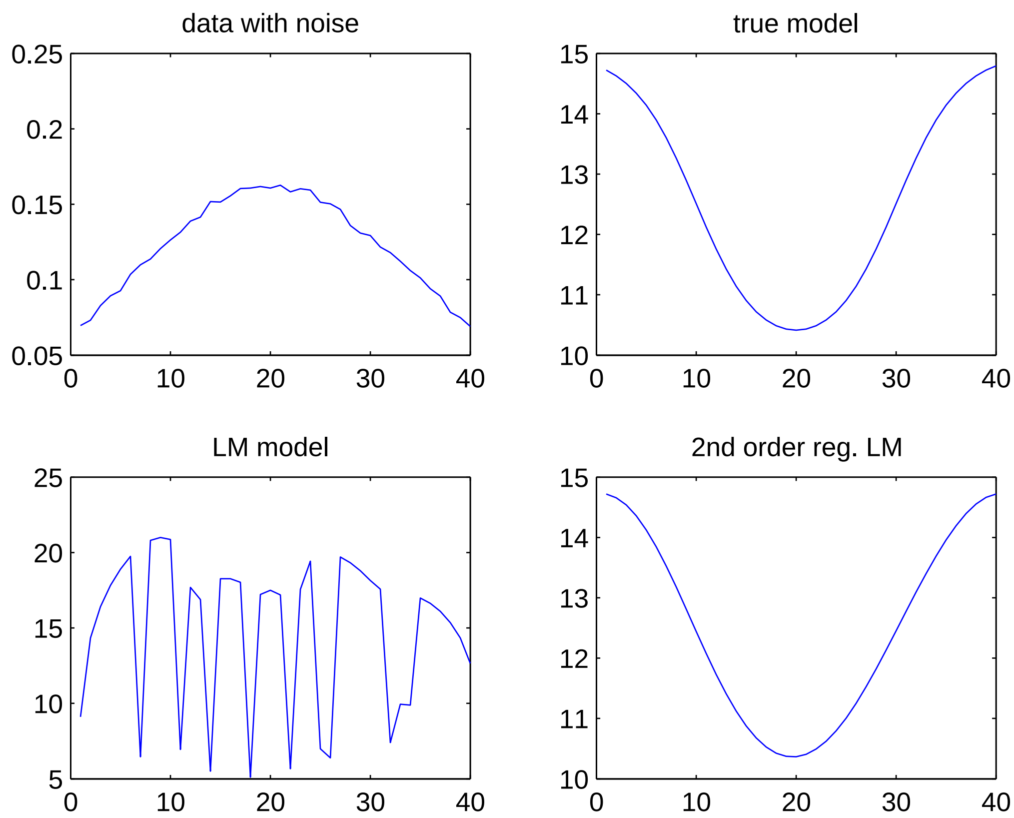
scientific method



gravity anomaly



$$W(x) = \rho MG \int \frac{D(x')dx'}{(D^2 + (x - x')^2)^{3/2}}$$

inverse problems

$$\overbrace{G}^{\text{theory}}(\overbrace{m}^{\text{model}}) = \overbrace{d}^{\text{data+`noise'}}$$

- **discrete** $\underbrace{\mathbf{G}}_{\mathbb{R}^{n\times m}}\underbrace{\mathbf{m}}_{\mathbb{R}^{m}} = \underbrace{\mathbf{d}}_{\mathbb{R}^{n}}$ or continuous $\int G(\psi, x)m(x)dx = d(\psi)$
- **linear** or nonlinear
- i.e. convolution, Fourier transform, Abel transform, Radon transform

– existence, uniqueness, stability
– “Riemann-Lebesque Lemma”

inverse problems

- filtering (tomography, SAR, planetary, pulse decoding)
- length methods (ISR lag profile analysis)
- MAP methods (Abel inversion, radar imaging)

These are equivalent!

filtering

$$Gm = d$$

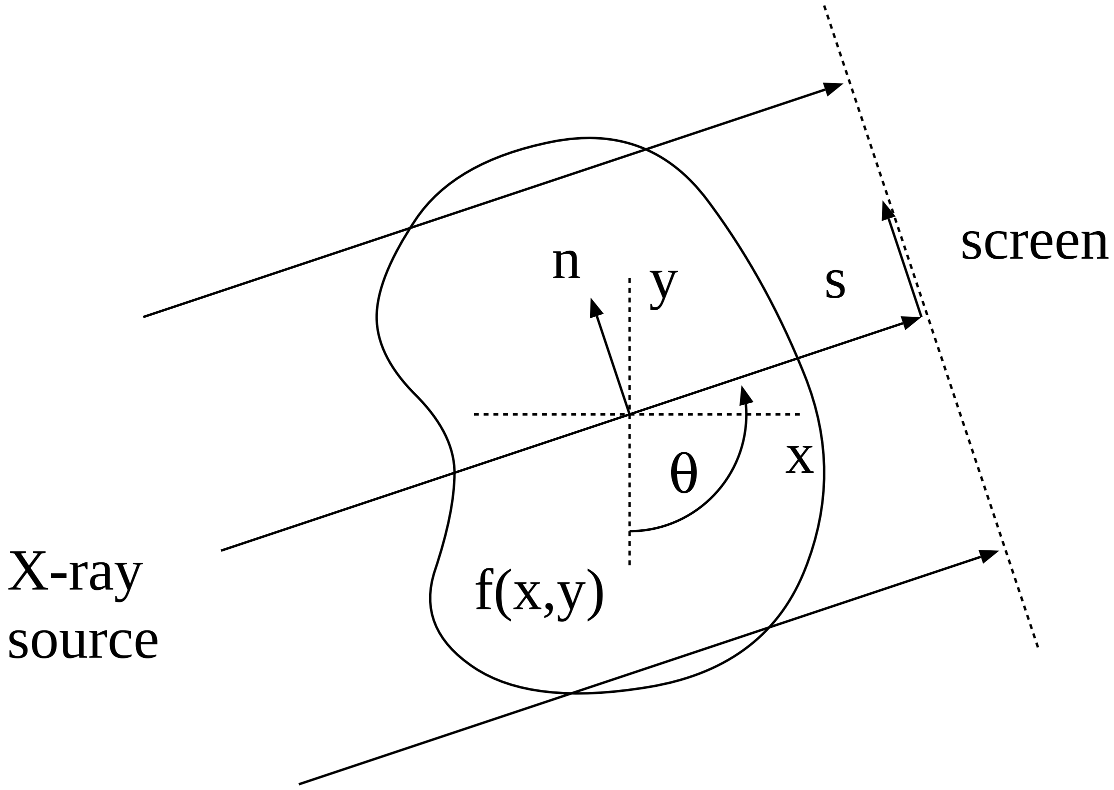
$$m = G^{\#}d = \underbrace{G^{\#}G}_{R_m = I?} m$$

$$Gm = \underbrace{GG^{\#}}_{R_d = I?} d = d$$

$$m(\xi) = \int dx \underbrace{\int d\psi G^{\#}(\xi,\psi)G(\psi,x)}_{K(\xi,x)=\delta(\xi-x)?} m(x)$$

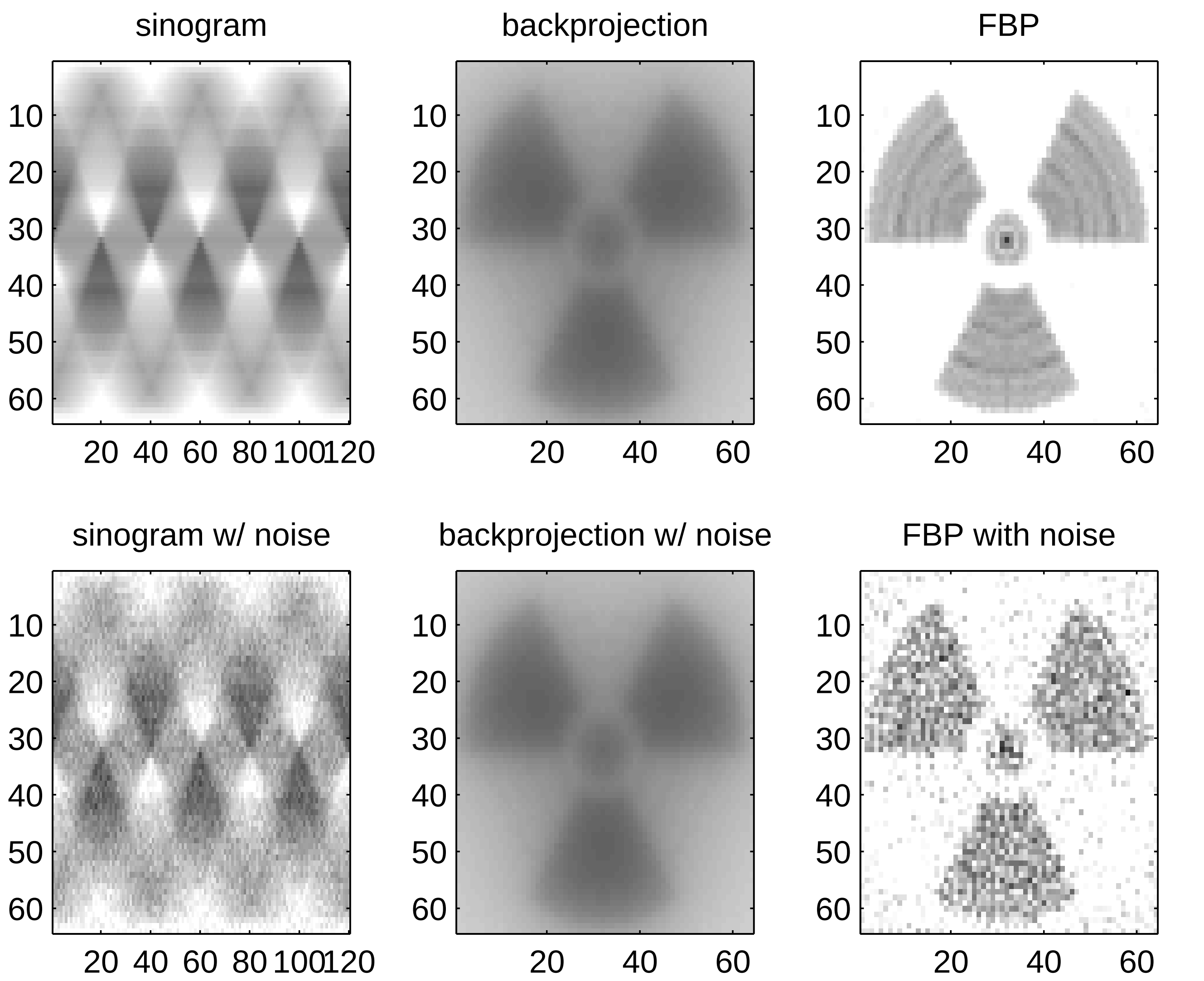
Backus Gilbert, filtered backprojection
... could be matched filtering, but probably isn't.

Radon transform



$$
\begin{aligned}
d(\theta, s) &= \int\int m(x,y)\delta(x\cos\theta + y\sin\theta - s)dxdy \\
\hat{d}(\theta, k) &= \hat{\hat{m}}(k\hat{n}_\theta)
\end{aligned}
$$

sinogram


sinogram
backprojection
FBP
sinogram w/ noise
backprojection w/ noise
FBP with noise
10
20
30
40
50
60
20 40 60 80 100 120
20 40 60

(filtered) backprojection

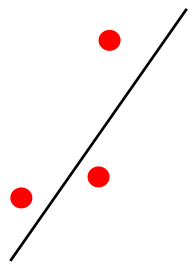
$$m(\mathbf{x}) \stackrel{?}{=} \int_0^{2\pi} d(\theta, s = \hat{n}(\theta) \cdot \mathbf{x}) d\theta \quad \text{adjoint}$$

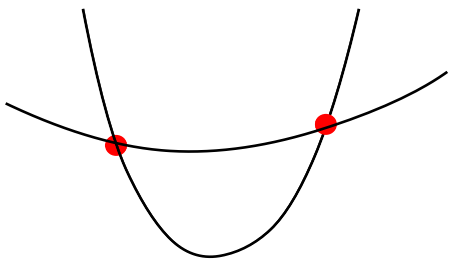
$$\begin{aligned}
H(k) &= |k| \quad \text{filter} \\
m(\mathbf{x}) &\stackrel{?}{=} \frac{1}{4\pi} \int_0^{2\pi} H d(\theta, s) d\theta \\
&= \frac{1}{4\pi} \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} |k| \hat{d}(\theta, k) e^{iks} dk d\theta \\
&= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{\infty} k \hat{d}(\theta, k) e^{ik(x\cos\theta + y\sin\theta)} dk d\theta \\
&= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{\infty} k \hat{\hat{m}}(k \hat{n}_\theta) e^{i\mathbf{k}\cdot\mathbf{x}} dk d\theta
\end{aligned}$$

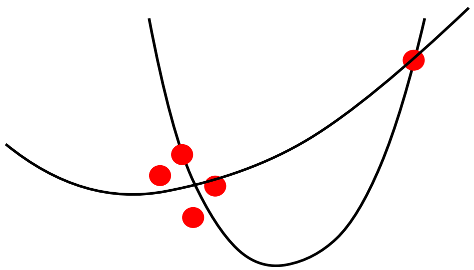
length methods

$Gm = d$, $G \epsilon \mathbb{R}^{nxm}$, rank[$G$]=$p$

	least squares	minimum length	weighted damped least squares
rank	$p = m < n$	$p = n < m$	$p < n, m$
termed	overdetermined	underdetermined	mixed determined
means?	no exact soln	no unique soln	mult equiv soln
min.	$(Gm - d)^t C_d^{-1}(Gm - d)$	$m^t C_m^{-1} m$	$e^t C_d^{-1} e + \alpha^2 m^t C_m^{-1} m$
m_{est}	$[G^t C_d^{-1} G]^{-1} G^t C_d^{-1} d$	$C_m^{-1} G^t [G C_m^{-1} G^t]^{-1} d$	$[G^t C_d^{-1} G + \alpha^2 C_m^{-1}]^{-1} G^t C_d^{-1} d$
			$C_m^{-1} G^t [G C_m^{-1} G^t + \alpha^2 C_d^{-1}]^{-1} d$
	max likelihood	Occam's razor	0, 1, 2 regularization







Moore Penrose pseudoinverse: existence, uniqueness, stability

$$
\begin{aligned}
G &= U\Lambda V^t, \quad Gm = d \quad Gx = 0 \quad x^t G = 0 \\
&= \underbrace{\left(\begin{array}{c|c} \text{column} & \text{left} \\ \text{space} & \text{nullspace} \end{array}\right)}_{nxn} \underbrace{\begin{pmatrix} \Lambda_{pxp} & 0 \\ 0 & 0 \end{pmatrix}}_{nxm} \underbrace{\begin{pmatrix} \text{rowspace} \\ \hline \text{nullspace} \end{pmatrix}}_{mxm} \\
G^\dagger &= V\Lambda^{-1}U^t, \quad m = G^\dagger d \\
&= \underbrace{\begin{pmatrix} V_{mxp} & V_{mx(m-p)} \end{pmatrix}}_{mxm} \underbrace{\begin{pmatrix} \Lambda^{-1}_{pxp} & 0 \\ 0 & 0 \end{pmatrix}}_{mxn} \underbrace{\begin{pmatrix} U^t_{pxn} \\ U^t_{(n-p)xn} \end{pmatrix}}_{nxn} \\
&= V_{mxp}\Lambda^{-1}_{pxp}U^t_{pxn}
\end{aligned}
$$

– condition no. $\equiv \Lambda_{\max}/\Lambda_{\min}$

quadratic regularization

The following are equivalent:

- gSVD using filter factors of the form $f_i = (s_i^2 + \alpha^2)/s_i^2$, where s_i are the singular values and α is the so-called regularization parameter. The data covariance matrix can be incorporated by transforming and scaling G and d.
- Minimization of the cost function $(Gm-d)^t C_d^{-1}(Gm-d) + \alpha^2 m^t C_m^{-1} m$, where $C_m^{-1} = L^t L$. The model estimator that accomplishes this is $m^{\mathrm{est}} = (G^t C_d^{-1} G + \alpha^2 C_m^{-1})^{-1} G^t C_d^{-1} d$ This strategy is termed 'weighted damped least squares.'
- Conjugate gradient weighted least squares minimization with an initial guess $m^{\mathrm{est}} = 0$ and with early iteration termination consistent with some finite α.

quadratic regularization II

- Augmented least squares, seeking the least-squares solution to the augmented minimization problem:

$$\min \left\| \begin{pmatrix} C_d^{-1/2} G \\ \alpha L \end{pmatrix} m - \begin{pmatrix} C_d^{-1/2} d \\ 0 \end{pmatrix} \right\|_2^2$$

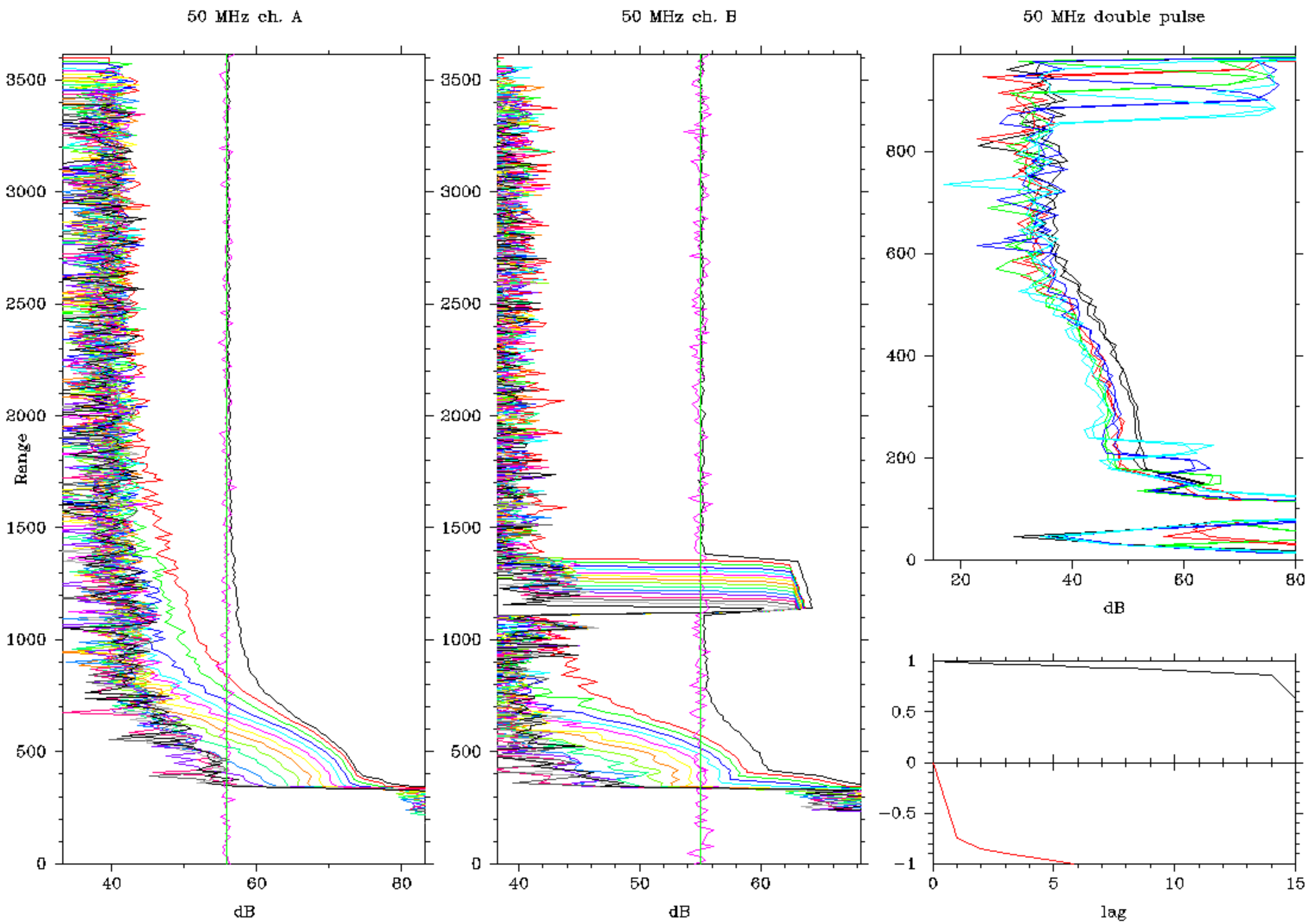
where $C_d^{-1/2t} C_d^{-1/2} = C_d^{-1}$.

- Solving (for m^{est}) the characteristic equation

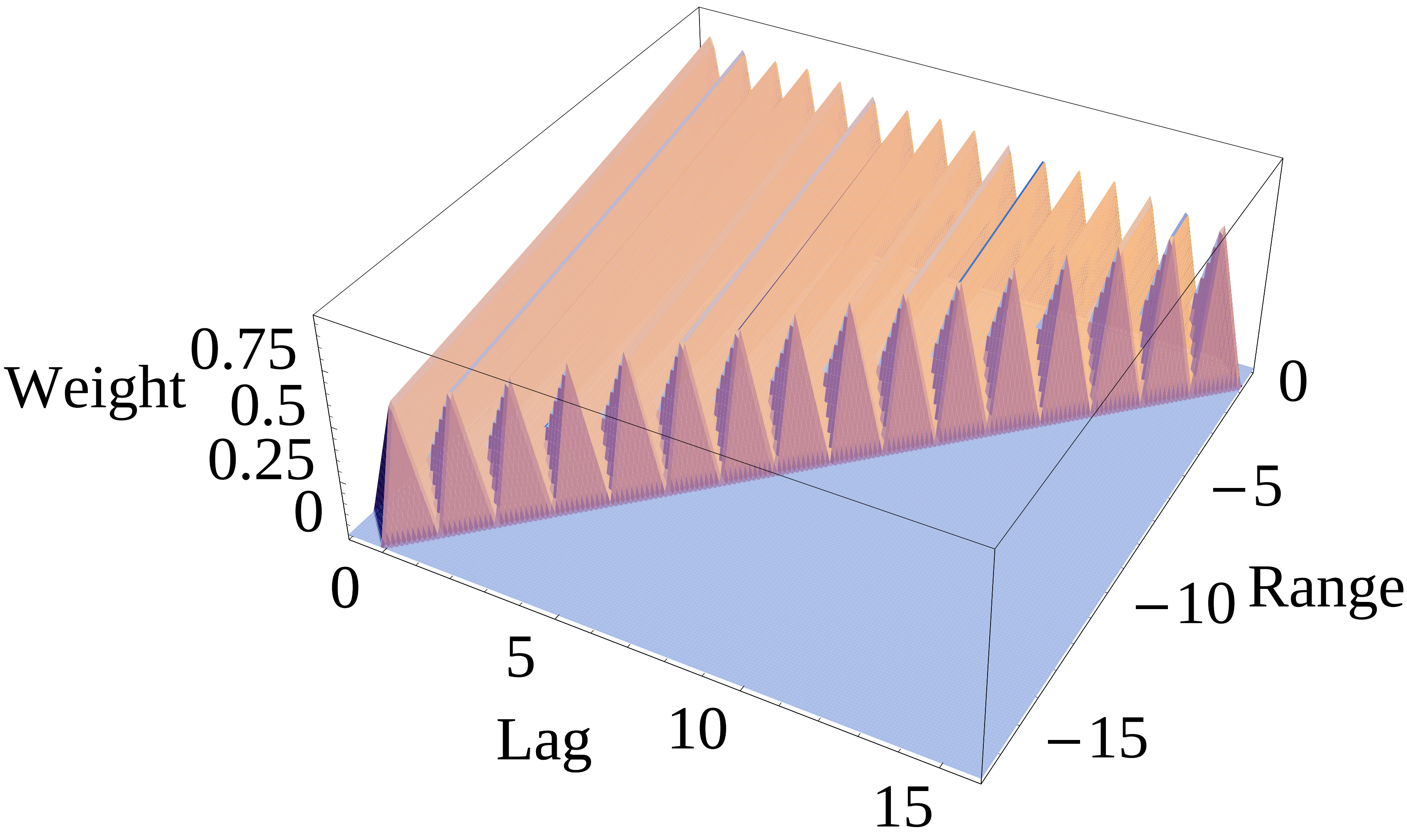
$$\begin{aligned} \begin{pmatrix} G^t C_d^{-1} & \alpha L^t \end{pmatrix} \begin{pmatrix} G \\ \alpha L \end{pmatrix} m \\ = \quad \begin{pmatrix} G^t C_d^{-1} & \alpha L^t \end{pmatrix} \begin{pmatrix} d \\ 0 \end{pmatrix} \end{aligned}$$

the result being the weighted damped least squares estimator above.

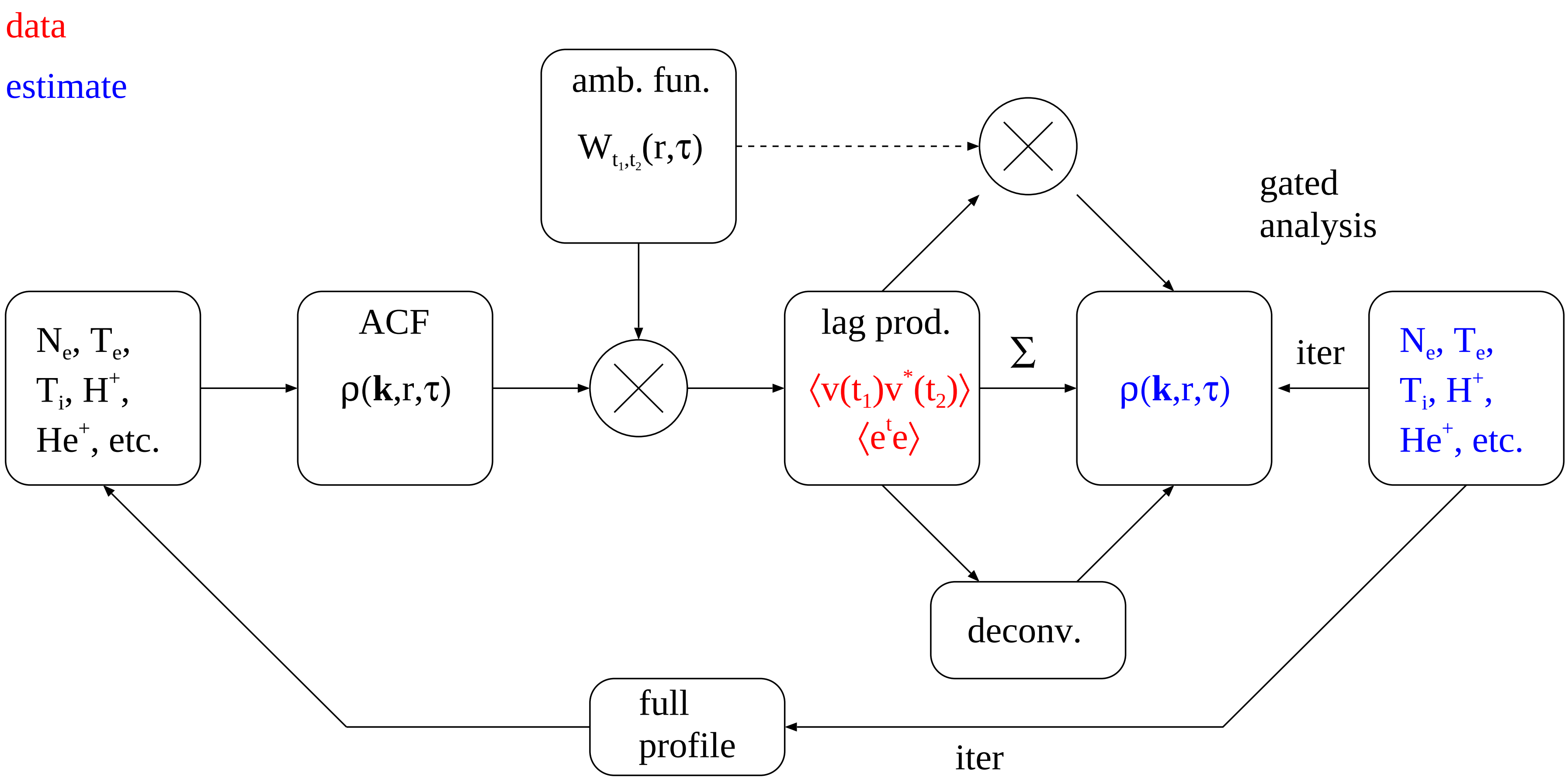
long-pulse data



ambiguity functions



full profile analysis

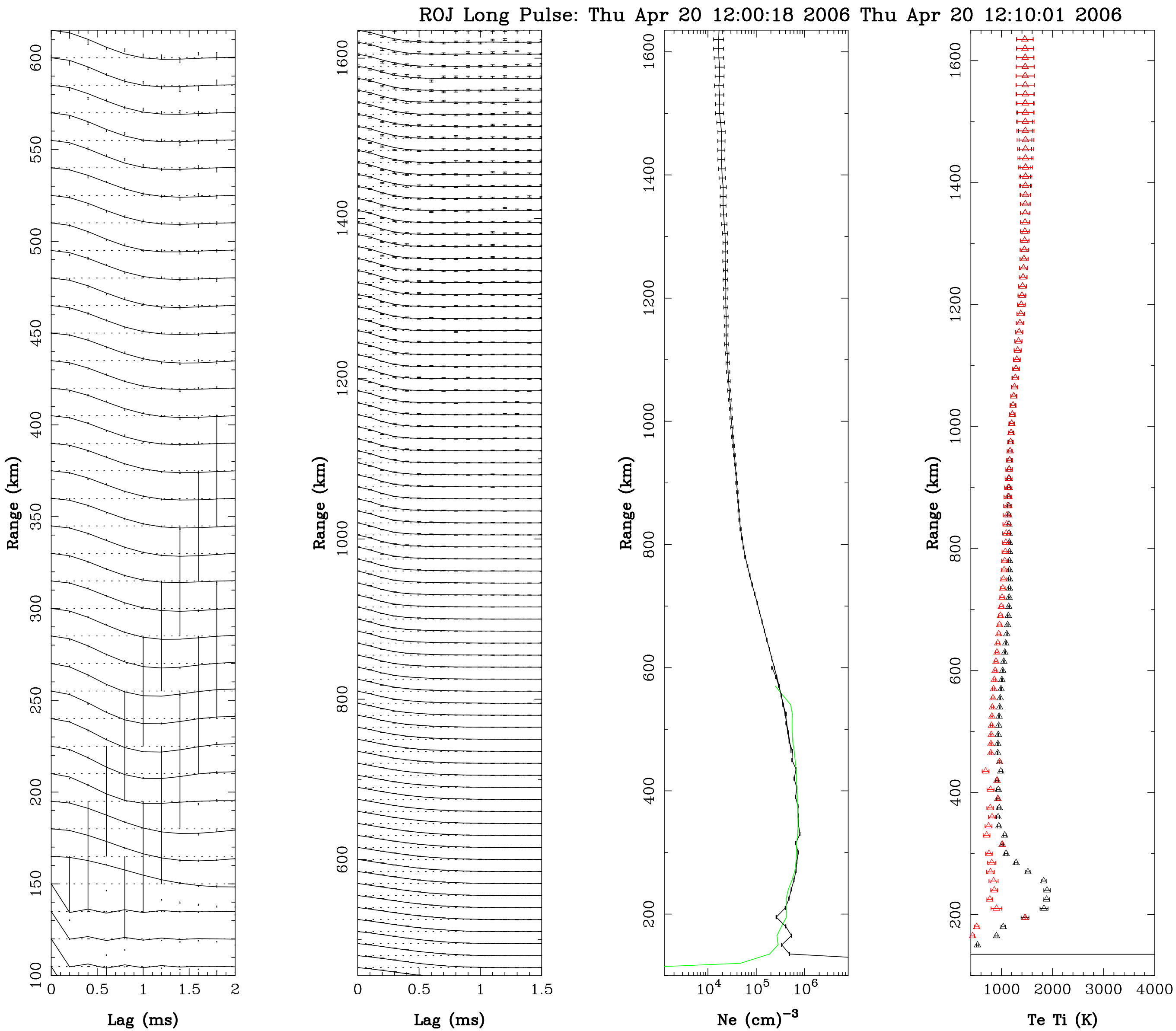


cost function

Augmented nonlinear least squares

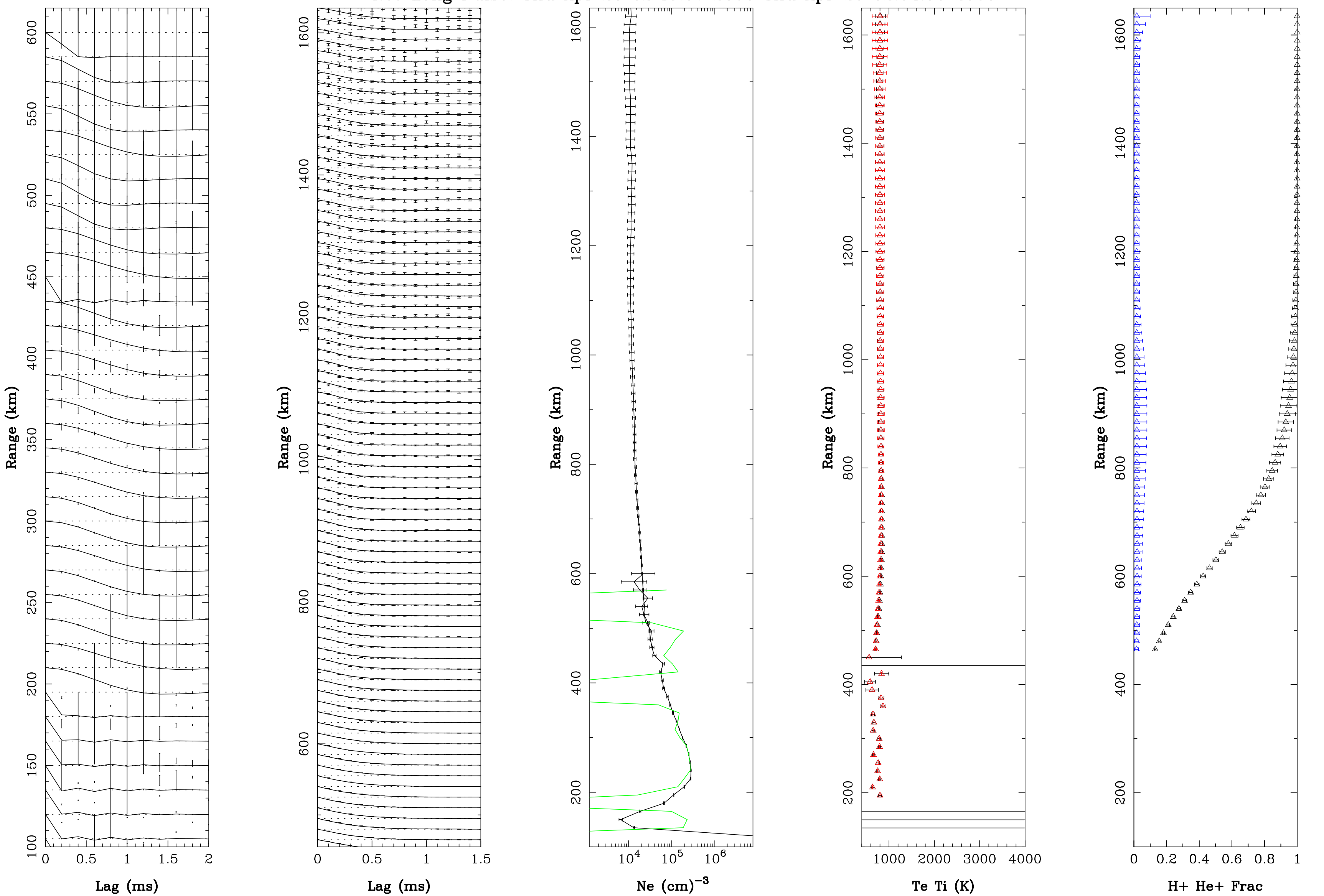
- prediction error norm $e^t C_d^{-1} e$
- $||T_e''||_2^2$ $||T_i''||_2^2$ temperature roughness
- $T_i/T_e \leq 1$ temperature ratio
- $||H^{+''}||_2^2$ hydrogen ion roughness
- composition fractions [0,1]

12 LT


ROJ Long Pulse: Thu Apr 20 12:00:18 2006 Thu Apr 20 12:10:01 2006
Range (km)
Lag (ms)
Ne (cm)$^{-3}$
Te Ti (K)
H+ He+ Frac

3 LT

ROJ Long Pulse: Thu Apr 20 02:45:07 2006 Thu Apr 20 02:54:50 2006

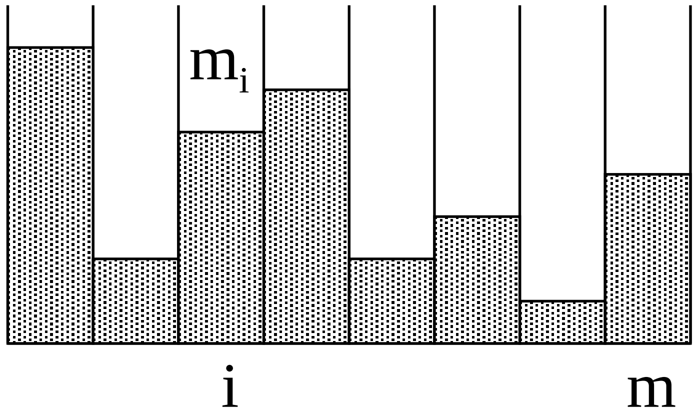


MAP methods, Bayes' theorem

$$P(m|d) = \frac{P(d|m)P(m)}{P(d)}$$

$$P(d|m) = \frac{1}{(2\pi)^{N/2}|C_d|^{1/2}} e^{-\frac{1}{2}(Gm-d)^t C_d^{-1}(Gm-d)}$$

$$P(m) = ?,\ e^{\alpha S}$$



$$M = \sum_{i=1}^{m} m_i$$

$$\text{perm} = \frac{M!}{\Pi_{i=1}^{m} m_i!}$$

$$S = -\sum_{i=1}^{m} m_i \log(m_i/M)$$

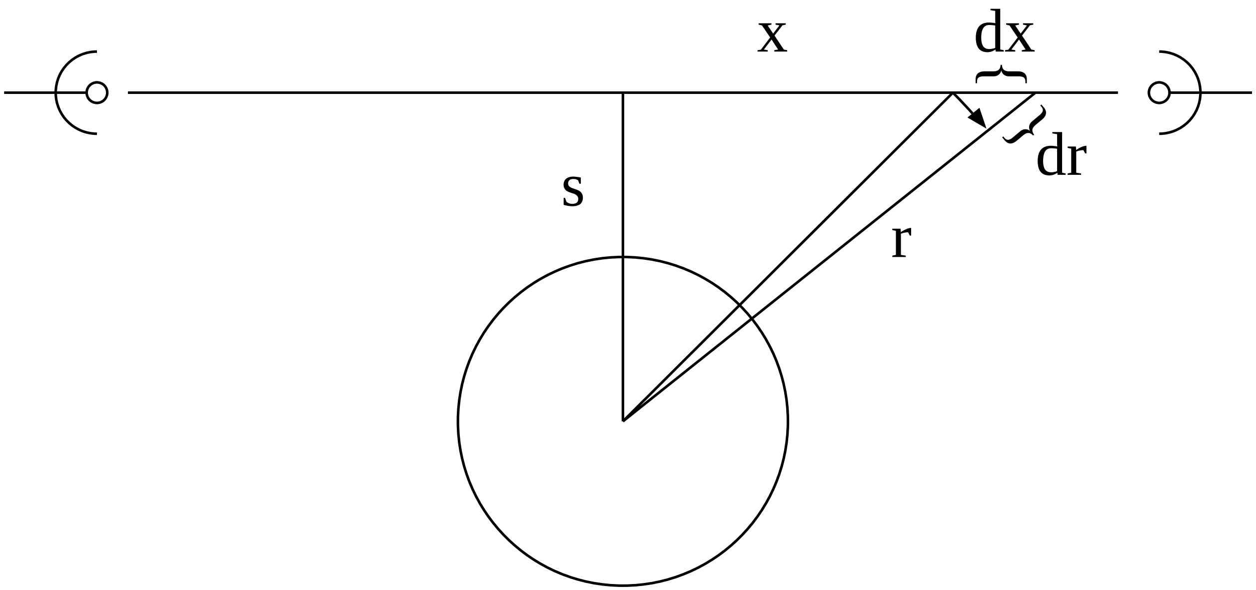
maximum entropy

$$E = S + \lambda^t(d + e - Gm) + \Lambda(e^t C_d^{-1} e - \Sigma)$$

$$\begin{aligned} m_i &= M\frac{e^{-\lambda^t G^{[,i]}}}{Z} \\ Z &= \frac{\hat{I}^t m}{M} \end{aligned}$$

$$E = \lambda^t(d + e) + M \log Z + \Lambda(e^t C_d^{-1} e - \Sigma)$$

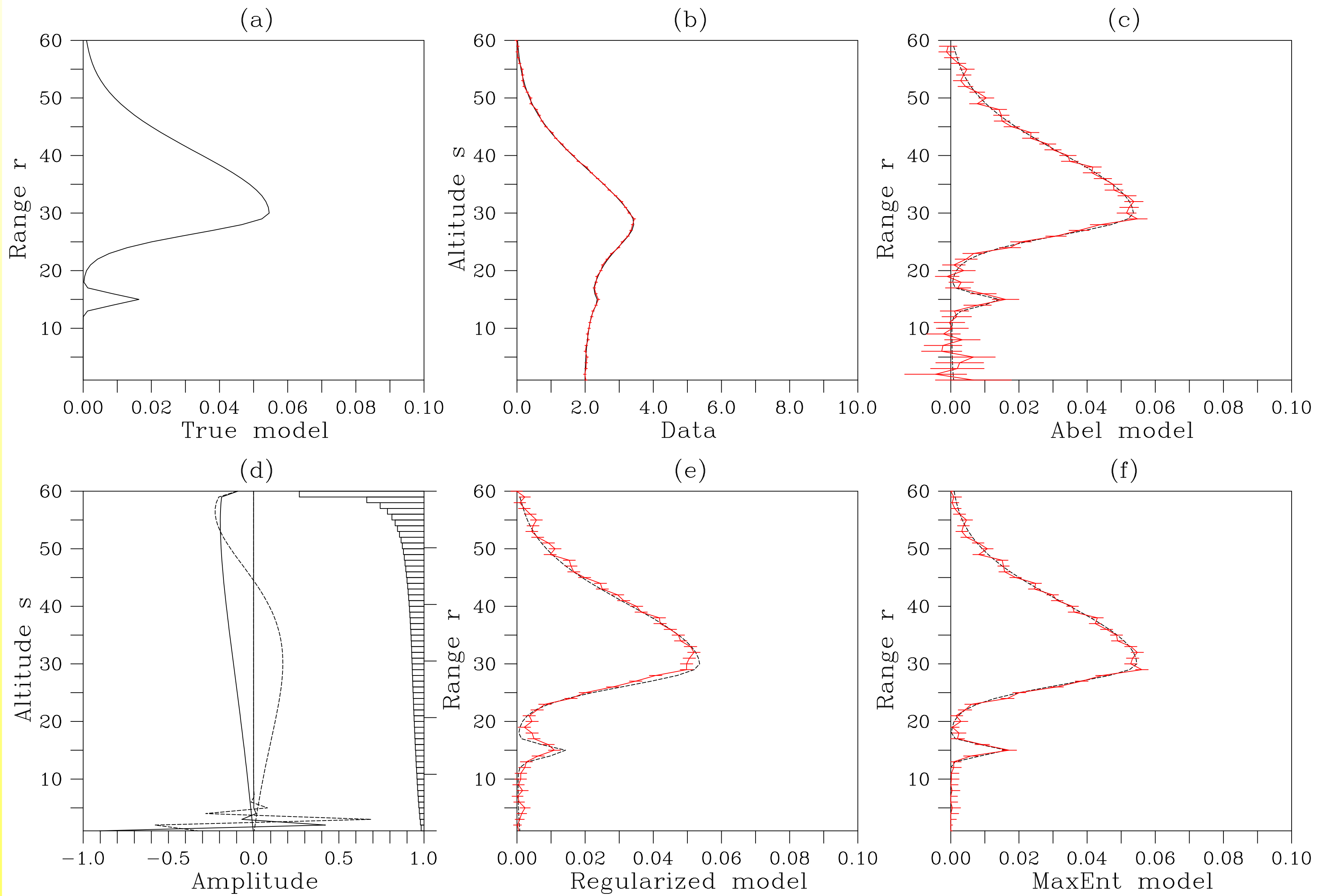
Abel transform



$$\phi(s) = 2C \int_s^\infty n_e(r) \frac{r dr}{\sqrt{r^2 - s^2}}$$

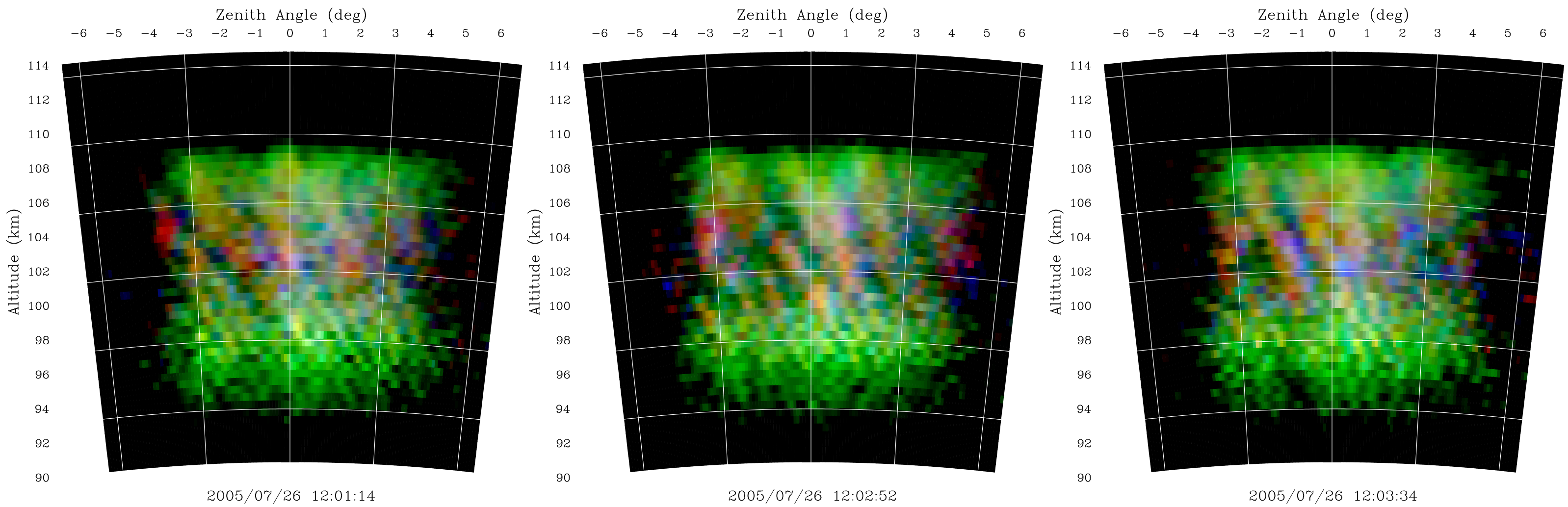
$$n_e(r) = -\frac{1}{\pi C} \int_r^\infty \frac{d\phi}{ds} \frac{ds}{\sqrt{s^2 - r^2}}$$

occultation simulation



not discussed

- regularization parameters (L-curve, GCV, adaptive methods)
- error analysis (full error covariance matrix)
- stability, speed, tradeoffs



references

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