JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 114, A07302, doi:10.1029/2009JA014200, 2009

Click Here for Full Article

Particle dynamics description of "BGK collisions" as a Poisson process

Marco Milla[1] and Erhan Kudeki[1]

Received 2 March 2009; revised 7 April 2009; accepted 14 April 2009; published 2 July 2009.

[1] The Gordeyev integral for plasma particles colliding with neutrals is obtained using a particle dynamics formalism in which the collisions are modeled as a discrete Poisson process. The result leads to an electron density fluctuation spectrum model for partially ionized plasmas which is identical with the spectral model obtained from BGK plasma kinetic equations. This isomorphism between the Poisson process and the BGK operator is analogous to a similar relation between the Brownian motion process and the Fokker-Planck operator with constant coefficients. We take advantage of this analogy to derive a collisional ISR spectrum model that takes into account collisions with both neutrals and charged species.

Citation: Milla, M., and E. Kudeki (2009), Particle dynamics description of "BGK collisions" as a Poisson process, *J. Geophys. Res.*, *114*, A07302, doi:10.1029/2009JA014200.

1. Introduction

[2] The purpose of this note is to show that incoherent scatter spectrum theory derived from the Boltzmann equation with BGK collision operator can also be obtained by using a particle dynamics approach where particle collisions are modeled as a Poisson process. This is analogous to Fokker-Planck operator (with constant coefficients) and the Brownian motion (Ornstein-Uhlenbeck) collision process leading to identical spectral models when utilized in the Boltzmann equation and in particle dynamics formalism, respectively [e.g., *Chandrasekhar*, 1943; *Gillespie*, 1996; M. Milla and E. Kudeki, Simulations of electron and ion Coulomb collisions in a magnetized plasma for ISR applications, poster presented at 2006 CEDAR Workshop, National Center for Atmospheric Research, Santa Fe, New Mexico, 2006].

[3] In general, incoherent scatter spectral theories pertinent to various types of ionospheric plasmas (magnetized, collisional, etc.) in thermal equilibrium can be expressed in terms of appropriately derived Gordeyev integrals utilized within a general framework described by *Kudeki and Milla* [2006]. Briefly, in this framework, the electron density $\mathbf{k}-\omega$ spectrum causing the incoherent radar scatter can be expressed (for a single ionic species, singly ionized; generalization to multiple ions is straightforward) as

$$\left\langle |n(\mathbf{k},\omega)|^2 \right\rangle = \frac{|j\omega\epsilon_o + \sigma_i|^2 \left\langle |n_{te}|^2 \right\rangle + |\sigma_e|^2 \left\langle |n_{ti}|^2 \right\rangle}{|j\omega\epsilon_o + \sigma_e + \sigma_i|^2}, \qquad (1)$$

with

$$\frac{\left\langle |n_{ts}(\mathbf{k},\omega)|^2 \right\rangle}{N_o} \equiv 2\mathrm{Re}\{J_s(\omega_s)\} \qquad (2)$$

and

$$\frac{\sigma_s(\mathbf{k},\omega)}{j\omega\epsilon_o} \equiv \frac{1 - j\omega_s J_s(\omega_s)}{k^2 h_s^2}, \qquad (3)$$

where $\omega_s \equiv \omega - \mathbf{k} \cdot \mathbf{V}_s$ is Doppler-shifted frequency due to mean velocity $\mathbf{V}_s$ of species s ("e" or "i" in the single-ion case) in the radar reference frame, $h_s = \sqrt{\epsilon_o K T_s / N_o e^2}$ is the corresponding Debye length, T_s the species temperature, N_o the mean plasma density, K Boltzmann constant, ϵ_o permittivity of free space, $-e$ electronic charge, and

$$J_s(\omega) \equiv \int_0^\infty d\tau \, e^{-j\omega\tau} \left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s} \right\rangle \qquad (4)$$

is a Gordeyev integral (a one sided Fourier transform) expressed in terms of characteristic function $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\rangle$ of random particle displacement vector $\Delta\mathbf{r}_s$ for species s over intervals τ. Different types of plasmas are distinguished by different types of $\Delta\mathbf{r}_s$ statistics, the specification of which, depending on the physical processes governing individual particle motions (with the exception of collective interactions), determines the species conductivities $\sigma_s(\mathbf{k}, \omega)$ and the corresponding spectra, $\langle |n_{ts}(\mathbf{k}, \omega)|^2\rangle$, of thermally impressed density fluctuations which "collectively drive" the observed electron density spectrum $\langle |n(\mathbf{k}, \omega)|^2\rangle$. The characteristic function $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\rangle$, an expected value that depends on the pdf of displacements $\Delta\mathbf{r}_s$, will also be termed as single particle signal correlation (or ACF), since signal return from a single particle exposed to a radar

[1]Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA.

Copyright 2009 by the American Geophysical Union.
0148-0227/09/2009JA014200$09.00

pulse would be proportional to $e^{j\mathbf{k}\cdot\mathbf{r}_s}$, with $\mathbf{r}_s = \mathbf{r}_s(t)$ denoting the particle trajectory.

[4] During the review process of *Kudeki and Milla* [2006], the validity of the general framework outlined above was initially questioned by one of the reviewers, who was concerned about BGK-based incoherent scatter spectral models being an exception to the proposed framework. Although the reviewer was ultimately shown that BGK-based models constitute no exception, i.e., they can be represented in terms of a Gordeyev integral corresponding to a one-sided Fourier transform of a single particle ACF, an explicit discussion of this was not included in the published version of the paper. In this note, we are revisiting the case of BGK-based incoherent scatter theories to provide an explicit derivation of the BGK density spectrum from a particle dynamics point of view. In particular, the BGK-based Gordeyev integral is obtained, in section 2, by assuming a Poisson collision process, and it is shown that its inclusion in the above framework yields the usual BGK model results [e.g., *Dougherty*, 1963; *Dougherty and Farley*, 1963] for the electron density spectrum. The paper is concluded with further discussions and implications of our results in section 3.

2. Derivation of the BGK Gordeyev Integral Using a Particle Dynamics Approach

[5] *Clemmow and Dougherty* [1969] state that the BGK model can be used to simulate the effect of binary collisions, e.g., collisions between charged and neutral particles, and also that these collisions could be imagined to be a Poisson process. However, this explanation was given more to provide a possible interpretation of the BGK model than to establish a direct relationship with the Poisson collision process. The mathematical proof that these two collision models are directly linked is provided next.

[6] Assume that in a time interval τ, a particle (electron or ion) collides n times with neutral particles constituting a medium. Collision events are independent from each other. In between collisions particles move in straight line orbits. We can then express the particle displacement over interval τ as the random vector

$$\Delta\mathbf{r}_s = \sum_{l=0}^{n} \mathbf{v}_l (t_{l+1} - t_l), \tag{5}$$

where $t_0 \equiv 0$, $t_{n+1} \equiv \tau$, and t_l is the time of lth collision such that $0 < t_1 < \ldots < t_n < \tau$.

[7] Let the number of collision events n invoked above be a Poisson random variable with a collision frequency ν and pmf

$$p(n) = e^{-\nu\tau}\frac{(\nu\tau)^n}{n!} \quad \text{for } n \geq 0. \tag{6}$$

Given that there are n collision events in an interval τ, the conditional pdf of collision times t_l (for $1 \leq l \leq n$) is given by [e.g., *Hajek*, 2009]

$$f(t_1, .., t_n|n) = \begin{cases} \dfrac{n!}{\tau^n} & \text{if } 0 < t_1 < \ldots < t_n < \tau \\ 0 & \text{else.} \end{cases} \tag{7}$$

This is in effect a uniform distribution over all ordered sets of collision times $0 < t_1 < \ldots < t_n < \tau$, that can also be written as

$$f(t_1, .., t_n|n) = \frac{n!}{\tau^n}\prod_{l=0}^{n} u(t_{l+1} - t_l) \tag{8}$$

in terms of unit-step function $u(t)$ and it satisfies $\int dt_n \ldots \int dt_1 f(t_1, .., t_n|n) = 1$ as all pdfs do. Additionally, assume that particle velocities $\mathbf{v}_l$, $l \in [0, n]$, in between the collisions constitute a set of independent and identically distributed random variables. Taking the distribution of vector velocities $\mathbf{v}_l$ as Maxwellian, we have

$$f(\mathbf{v}_l) = \frac{e^{-\frac{1}{2C^2}v_l^2}}{(2\pi C^2)^{3/2}}, \tag{9}$$

where $C = \sqrt{\frac{KT}{m}}$ is the thermal speed of the particles.

[8] Let us next compute the single particle ACF $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\rangle$ (the characteristic function of displacements $\Delta\mathbf{r}_s$) required in the general framework outlined above, where the expected value will be computed over random variables $\mathbf{v}_0, \ldots, \mathbf{v}_n, t_1, \ldots, t_n$, and n using the probability distributions defined above. We note that

$$\left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\right\rangle = \left\langle e^{j\mathbf{k}\cdot\sum_{l=0}^{n}\mathbf{v}_l(t_{l+1}-t_l)}\right\rangle_{\mathbf{v},t,n} = \left\langle \prod_{l=0}^{n} e^{j\mathbf{k}\cdot\mathbf{v}_l(t_{l+1}-t_l)}\right\rangle_{\mathbf{v},t,n}, \tag{10}$$

where the subscripts on the right indicate successive expected value operations to be performed. Starting with the expectations over independent Gaussian random variables $\mathbf{v}_l$, we have

$$\left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\right\rangle = \left\langle \prod_{l=0}^{n} e^{-\frac{1}{2}k^2C^2(t_{l+1}-t_l)^2}\right\rangle_{t,n}. \tag{11}$$

Expectations over $t_1, \ldots, t_n$ next yield

$$\left\langle \frac{n!}{\tau^n}\int dt_n \cdots \int dt_1 \prod_{l=0}^{n} e^{-\frac{1}{2}k^2C^2(t_{l+1}-t_l)^2} u(t_{l+1} - t_l)\right\rangle_n, \tag{12}$$

where integration limits run from $-\infty$ to ∞. Finally, using $p(n)$, we find that the ACF is

$$\sum_{n=0}^{\infty} \nu^n e^{-\nu\tau}\int dt_n \cdots \int dt_1 \prod_{l=0}^{n} e^{-\frac{1}{2}k^2C^2(t_{l+1}-t_l)^2} u(t_{l+1} - t_l). \tag{13}$$

Now we define

$$g(t) \equiv e^{-\nu t - \frac{1}{2}k^2C^2t^2}u(t), \tag{14}$$

to rearrange (13) as

$$\begin{aligned} \left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\right\rangle &= \sum_{n=0}^{\infty} \nu^n \int dt_n \cdots \int dt_1 \prod_{l=0}^{n} g(t_{l+1} - t_l) \\ &= \sum_{n=0}^{\infty} \nu^n \int dt_n g(\tau - t_n) \cdots \int dt_1 g(t_2 - t_1) g(t_1). \end{aligned} \tag{15}$$

Clearly, the integral chain above is n successive convolutions of $n + 1$ realizations of $g(t)$ evaluated at $t = \tau$. Thus, the ACF of particles with a Maxwellian velocity distribution undergoing a Poisson collision process reduces to

$$\left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\right\rangle = \sum_{n=0}^{\infty} \nu^n \underbrace{g(\tau) * \cdots * g(\tau)}_{n+1}. \qquad (16)$$

Since $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s}\rangle$ is the characteristic function of $\Delta\mathbf{r}_s$, i.e., the Fourier transform of the pdf of $\Delta\mathbf{r}_s$, our particle displacement model with Poisson collisions is uniquely described by (16).

[9] The corresponding Gordeyev integral (4), i.e., one-sided Fourier transform of the single-particle ACF (16), is then (using the properties of the convolution and the Fourier transform)

$$J_s(\omega) = G(\omega)\sum_{n=0}^{\infty} \nu^n G^n(\omega), \qquad (17)$$

where

$$G(\omega) \equiv \int_0^{\infty} d\tau e^{-j\omega\tau} e^{-\nu\tau - \frac{1}{2}k^2C^2\tau^2} \qquad (18)$$

is the Fourier transform of $g(t)$. The convergence of the series in (17) is guaranteed as

$$\sum_{n=0}^{\infty} \nu^n G^n(\omega) = \frac{1}{1 - \nu G(\omega)} \qquad (19)$$

since $|\nu G(\omega)| < 1$ for any $\nu \geq 0$. Hence, Gordeyev integral (17) reduces to

$$J_s(\omega) = \frac{G(\omega)}{1 - \nu G(\omega)}, \qquad (20)$$

a well-known result previously derived by *Dougherty* [1963] using the BGK collision model for unmagnetized plasmas.

3. Discussion

[10] We can argue, on the basis of our result in section 2, that from a particle dynamics perspective, the BGK collision process is a Poisson process. As a consequence, any of the expected quantities that can be obtained using the BGK kinetic equation, can also be derived using our stochastic model for particle dynamics, for example, species conductivities $\sigma_s(\mathbf{k}, \omega)$ and the spectra of thermally impressed density fluctuations $\langle|n_{ts}(\mathbf{k}, \omega)|^2\rangle$.

[11] The general framework for incoherent scatter spectrum models presented by *Kudeki and Milla* [2006] can be developed independent of plasma kinetic equations, on the basis of only the following fundamental relations: the fluctuation-dissipation or Nyquist theorem that relates $\langle|n_{ts}(\mathbf{k}, \omega)|^2\rangle$ and $\mathrm{Re}\{\sigma_s(\mathbf{k}, \omega)\}$ for particles in thermal equilibrium [e.g., *Callen and Welton*, 1951], and the Kramers-Kronig relations that connect the real and imaginary parts of $\sigma_s(\mathbf{k}, \omega)$ in order to satisfy the principle of causality [e.g., *Clemmow and Dougherty*, 1969]. As we have shown above, plasmas can also be studied on the basis of these principles, and thus, whether we use kinetic equations or the particle dynamics approach is a matter of choice in solving plasma problems concerning particles in thermal equilibrium (including the phenomenon of Landau damping).

[12] Let us now calculate, as an example, the covariance matrix of particle displacements in a Poisson collision process. The covariance matrix is defined as

$$\left\langle \Delta\mathbf{r}_s\Delta\mathbf{r}_s^{\mathsf{T}}\right\rangle = \left\langle \sum_{l=0}^{n}\sum_{l'=0}^{n} \mathbf{v}_l\mathbf{v}_{l'}^{\mathsf{T}}(t_{l+1} - t_l)(t_{l'+1} - t_{l'})\right\rangle_{\mathbf{v},t,n}. \qquad (21)$$

Since $\mathbf{v}_l$ are independent random variables, we have

$$\left\langle \Delta\mathbf{r}_s\Delta\mathbf{r}_s^{\mathsf{T}}\right\rangle = \left\langle \sum_{l=0}^{n} \mathbf{v}_l\mathbf{v}_l^{\mathsf{T}}(t_{l+1} - t_l)^2\right\rangle_{\mathbf{v},t,n}. \qquad (22)$$

Furthermore, provided that the components of $\mathbf{v}_l$ are Gaussian and independent, the covariance matrix of $\mathbf{v}_l$ is given by

$$\left\langle \mathbf{v}_l\mathbf{v}_l^{\mathsf{T}}\right\rangle = C^2\bar{I}, \qquad (23)$$

where $\bar{I}$ is the identity matrix. Thus,

$$\left\langle \Delta\mathbf{r}_s\Delta\mathbf{r}_s^{\mathsf{T}}\right\rangle = C^2\bar{I}\left\langle \sum_{l=0}^{n} (t_{l+1} - t_l)^2\right\rangle_{t,n}. \qquad (24)$$

After some math, it can be verified that

$$\left\langle \sum_{l=0}^{n} (t_{l+1} - t_l)^2\right\rangle_{t} = \frac{2\tau^2}{n+2}. \qquad (25)$$

Taking the expected value of (25) with respect to n and substituting in (24) gives

$$\begin{aligned}\left\langle \Delta\mathbf{r}_s\Delta\mathbf{r}_s^{\mathsf{T}}\right\rangle &= C^2\bar{I}\sum_{n=0}^{\infty} e^{-\nu\tau}\frac{(\nu\tau)^n}{n!}\frac{2\tau^2}{n+2} \\ &= \frac{2C^2}{\nu^2}\bar{I}\sum_{n=0}^{\infty} e^{-\nu\tau}\frac{(n+1)(\nu\tau)^{n+2}}{(n+2)!},\end{aligned} \qquad (26)$$

which simplifies as

$$\left\langle \Delta\mathbf{r}_s\Delta\mathbf{r}_s^{\mathsf{T}}\right\rangle = \frac{2C^2}{\nu^2}\bar{I}(\nu\tau - 1 + e^{-\nu\tau}). \qquad (27)$$

[13] This is a very interesting result because the same mathematical expression can be derived in the context of a Brownian motion collision model [e.g., *Chandrasekhar*,

1943], an approach that it is often used to describe Coulomb collisions [e.g., *Zagorodny and Holod*, 2000]. In the Brownian motion formalism, the effects of collisions on particle motion are considered to be caused by the action of a friction force and random diffusive forces. A parameter analogous to ν is also defined, but it is regarded as a friction coefficient. Although, the expressions for $\langle \Delta \mathbf{r}_s \Delta \mathbf{r}_s{}^{\mathsf{T}} \rangle$ are the same for the Poisson and the Brownian motion models, the corresponding expressions for the single particle ACFs are not equal and lead to different incoherent scatter spectral shapes [e.g., *Hagfors and Brockelman*, 1971]. The differences, however, are only noticeable for intermediate values of ν since both spectra converge to the same asymptotic expressions in the collisionless/frictionless ($\nu \to 0$) as well as high collision/friction ($\nu \to \infty$) limits. The difference at intermediate ν can be attributed to $\Delta \mathbf{r}_s(\tau)$ being a Gaussian random variable at each τ in case of a Brownian motion process, but only so in $\tau \to 0$ and ∞ limits for a Poisson process; when $\Delta \mathbf{r}_s(\tau)$ is strictly Gaussian, and only then, the ACF $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s} \rangle$ can be shown to reduce to $e^{-\frac{1}{2}k^2\langle \Delta r_s^2 \rangle}$, where $\langle \Delta r_s^2 \rangle$ is a diagonal element of $\langle \Delta \mathbf{r}_s \Delta \mathbf{r}_s{}^{\mathsf{T}} \rangle$.

[14] A generalization of the results presented in section 2 can be easily performed. Notice that the assumption that in between collisions the particles move in straight line orbits was not a necessary condition and it was only considered in order to recover the classical results of the BGK collision model. In between collision events, we could have considered the particles moving in helical orbits due the action of an external magnetic force or even move randomly because of Coulomb interactions with other charged particles constituting a plasma. Either of these assumptions would have led to different definitions of the function $G(\omega)$, but it can be shown that the form of the Gordeyev integral $J_s(\omega)$, i.e., equation (17), would have remained the same. In general, it is found that

$$G(\omega) = \int_0^\infty d\tau e^{-j\omega\tau} e^{-\nu\tau} \left\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s^i} \right\rangle, \tag{28}$$

where $\langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}_s^i} \rangle$ is the single particle ACF of the process that takes place in between Poisson collision events. For instance, let us consider an unmagnetized plasma in which both neutral and Coulomb collisions are relevant. Modeling the neutral collisions as a Poisson process of frequency ν, and Coulomb collisions as a Brownian motion process, $G(\omega)$ takes the form

$$G(\omega) = \int_0^\infty d\tau e^{-j\omega\tau} e^{-\nu\tau} e^{-\frac{k^2 C^2}{\beta^2}\left(\beta\tau - 1 + e^{-\beta\tau}\right)}, \tag{29}$$

where we have used β to denote the friction coefficient in Brownian motion. This expression together with (17) provides us with a model for ionospheric incoherent scatter spectrum measurements detected from regions in which both neutral and Coulomb collisions are expected to be important, e.g., the 150-km region. More general extensions of the Poisson collision model can be pursued, for instance, a velocity-dependent collision frequency $\nu(\mathbf{v})$ could be considered into the theory. Further generalizations of this type will be the subject of future studies.

[15] **Acknowledgments.** This work has been funded by the NSF grant ATM-0215246 to the University of Illinois.

[16] Amitava Bhattacharjee thanks the reviewers for their assistance in evaluating this paper.

References

Callen, H. B., and T. A. Welton (1951), Irreversibility and generalized noise, *Phys. Rev.*, *83*(1), 34–40.

Chandrasekhar, S. (1943), Stochastic problems in physics and astronomy, *Rev. Mod. Phys.*, *15*(1), 1–89.

Clemmow, P. C., and J. P. Dougherty (1969), *Electrodynamics of Particles and Plasmas*, Addison-Wesley, Redwood City, Calif.

Dougherty, J. P. (1963), The conductivity of a partially ionized gas in alternating electric fields, *J. Fluid Mech.*, *16*(1), 126–137.

Dougherty, J. P., and D. T. Farley (1963), A theory of incoherent scattering of radio waves by a plasma: 3. Scattering in a partly ionized gas, *J. Geophys. Res.*, *68*, 5473–5486.

Gillespie, D. T. (1996), The multivariate Langevin and Fokker–Planck equations, *Am. J. Phys.*, *64*(10), 1246–1257.

Hagfors, T., and R. A. Brockelman (1971), A theory of collision dominated electron density fluctuations in a plasma with applications to incoherent scattering, *Phys. Fluids*, *14*(6), 1143–1151.

Hajek, B. (2009), *An Exploration of Random Processes for Engineers*, Dep. of Electr. and Comput. Eng., Univ. of Ill. at Urbana-Champaign, Urbana.

Kudeki, E., and M. Milla (2006), Incoherent scatter spectrum theory for modes propagating perpendicular to the geomagnetic field, *J. Geophys. Res.*, *111*, A06306, doi:10.1029/2005JA011546.

Zagorodny, A., and I. Holod (2000), Large-scale fluctuations and particle diffusion across external magnetic field in turbulent plasmas, *Condens. Matter Phys.*, *3*(2), 295–306.

E. Kudeki and M. Milla, Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, 1308 West Main, Urbana, IL 61801, USA. (erhan@illinois.edu; mmilla@illinois.edu)