

INCOHERENT SCATTERING OF ELECTROMAGNETIC
WAVES BY A PLASMA

A thesis presented

by

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SYNOPSIS

This thesis is concerned with the theory of incoherent scattering by a plasma, i.e., with the statistical description of the electromagnetic waves scattered by the thermal electron density fluctuations in a plasma which has been illuminated by an electromagnetic wave of certain frequency. We have treated the problem starting from first principles, taking a new approach and removing in the process some of the limitations inherent in other approaches taken so far for the solution of this problem. We have used this approach to solve, up to the inclusion of some numerical results, the pending problem of the effect of ion-ion "Coulomb collisions" on the ion gyro-resonances predicted by existent collisionless theories; this is an important aspect of the theory in connection with the application of incoherent scattering for the study of the ionosphere.

We can divide this paper into two main parts with different degrees of generality. The first part (Chapters II and III) is more formal and more general in scope, it is valid for non-homogeneous and non-stationary plasmas. In this first part we show that:

- (a) the time autocorrelation function of the scattered wave is a functional of the space-time electron density autocorrelation function of the plasma, and
- (b) that the latter in turn can be written as a product of two densities; the density

of the plasma in question and the electron density of the same plasma but under certain hypothetical initial conditions.

This reduces the problem to that of solving an initial value problem for the set of coupled kinetic equations of the plasma. In the second part (Chapters IV to VI) we carry out the solution of such an initial value problem but under less general conditions. We assume thermodynamic equilibrium and that the ion mean free path is larger than the scale sizes considered. These conditions are normally met in ionospheric applications. We use a simplified Fokker-Planck collision term, and show that under the above conditions only a few terms from a more complete model proposed by Dougherty are necessary. The effect of a constant magnetic field is included.

We present a computer technique used to obtain the time-autocorrelation of the scattered wave without computation of the frequency spectrum, avoiding in this way lengthy numerical Fourier transformations. We have evaluated in this way the autocorrelation function under a variety of conditions showing the effect of ion-ion Coulomb collisions, changes in ion composition and variations in the angle between the wave vector and the magnetic field. Using typical ionospheric parameters and wavelengths used by the Jicamarca Radio Observatory we conclude that they should be able to observe the gyro-frequency resonances for $[\text{He}]^+$ and $[\text{H}]^+$ under their normal concentrations, and the $[O]^+$ resonance only in those regions where the concentration is of the order of 10^4 ions/cm³ or less.

CHAPTER I

INTRODUCTION AND DEFINITIONS

1. Introduction.

When a plasma is illuminated by an electromagnetic wave, each electron is forced to oscillate at the incident frequency plus or minus an increment due to the Doppler shift caused by the thermal motion of the electrons. This oscillation in turn causes each of the electrons to re-radiate in all directions at a frequency equal to the forcing one corrected once more by the Doppler shift due to the velocity component in the scattered direction. It is only in the direction of the incident wave that the frequency of the scattered wave is equal to the incident frequency and adds up coherently to it. The dielectric properties of the plasma are a consequence of this "coherent scattering" of the electrons. In any other direction the scattered waves have a frequency which is different from the frequency of the incident wave by an amount which depends on the random velocity of the particular scattering electron and consequently add up incoherently, i.e., with a random (but not independent) phase. In this thesis we are mainly interested in the statistical description of this "incoherently scattered" waves and its relationship to the macroscopic properties of the medium such as density, composition, velocity distribution, temperature, etc.

There is already a considerable amount of theoretical work dedicated to this problem (references 1-18, for review papers, see 19-21), and the information contained in the frequency spectrum of the scattered wave is currently being used for the study of ionospheric plasmas using powerful radar observatories like the ones installed in Jicamarca, Peru; Millstone Hill, Massachusetts, U.S.A.; and Arecibo, Puerto Rico. Most of the theoretical work done so far has been motivated by this application and so is the work presented in this thesis.

One of the main conclusions of the theory is that the particle-to-particle correlations play an important role, especially the electron-to-ion correlation. Although the electrons are the ones that do the scattering, the spectrum of the scattered signal is in most cases determined by the dynamics of the ions. The spectrum of the signal has a characteristic width which corresponds to a Doppler shift of a thermal ion rather than of thermal electron as one would expect from pure electron Thomson-Rayleigh scattering (where the contribution of each electron has a random and independent phase). An important feature of the spectrum is the occurrence of resonant lines at approximately multiples of the ion gyro-frequencies. In terms of the corresponding autocorrelation function the gyro-resonances show as peaks at time delays approximately equal to multiples of the gyro-period. This feature of the spectrum is advantageous for ionospheric applications since it allows one to deduce ion composition from incoherent scatter observations.

Despite the extensive amount of theoretical work done on the problem of

incoherent scattering, this is still not complete. In the sense that, with the exemption of two papers, one by Farley⁵ and the other by Dougherty⁶ which we shall briefly discuss, no one has included the effect of ion-ion interaction or so-called Coulomb collisions. Ionospheric plasmas, even in their most dense regions (F_{max} peak), have an effective ion-ion collision frequency of the order of 7 sec^{-1} , which is small as compared to other characteristic frequencies like the ion gyro-frequency and the thermal characteristic frequency (inverse of the time it takes an ion to travel one wavelength). These frequencies are of the order of 160 and 2000 radians/second (for $\lambda = 1.5$ meters) respectively. Thus, initially it appeared that neglecting the effect of this type of collision was justified. A more careful analysis shows that this is not the case, especially in predicting the effects of the magnetic field and ion gyro-resonance phenomena. A more careful analysis was stimulated by the fact that $[0]^+$ ion-gyro-resonances predicted by the collisionless model were not observed experimentally.

In the paper by Farley⁵ mentioned above, he estimated the amount of probabilistic diffusion that any ion suffers after a gyro-period when subject to Coulomb collisions. From this he concluded, as we shall also see, that the effect of Coulomb collisions is indeed important and responsible for the failure of the experimental observation of oxygen gyro-resonances.

The other paper which considers the effect of Coulomb collisions is the one by Dougherty.⁶ He presents a Fokker-Planck type collision model for the Boltzmann equation and its analytical solution. He considers a single component

plasma and no self-consistent field. The incoherent scatter problem, regardless of the approach, requires the solution of Boltzmann-like equations, but with the inclusion of at least two species (ions and electrons) and the inclusion of the self-consistent field. So in regards to this problem this paper can be considered as an important step towards its solution, but yet not complete. By discussing his solution, he also demonstrates the importance of the collision term for typical ionospheric parameters.

We were motivated to study the incoherent scatter problem (and the associated one of plasma density fluctuations) because of the incompleteness of the solutions offered so far in regard to the effect of Coulomb collisions. Our final goal is to investigate the effect of such collisions, mainly on the ion-gyro-resonances predicted by the collisionless theory. But, the contributions presented in this thesis are not limited to the inclusion of the effects of such collisions in our solution. We present a solution to the problem taking a new approach, starting from first principles, and removing in the process some of the limitations of the other approaches taken so far. We present a technique which formally could be used even in the case of non-homogeneous, non-stationary plasmas. We have used this technique here to solve the important practical case of a homogeneous plasma in thermodynamical equilibrium (collisions included), with the inclusion of some numerical results. We also investigate numerically the effects of different ion composition and the direction of the magnetic field.

There are different steps in the solution of the incoherent scatter problem,

the task being: to obtain a statistical characterization of the scattered wave, i.e., its autocorrelation function,^(*) at any given location and as a function of those statistical parameters or functions which are sufficient to determine the state of the illuminated plasma.

The first step is taken in Chapter II where we show under very general conditions that the time autocorrelation function of the scattered wave field, or of the signal it produces in a (linear) receiving instrument, is given as a functional of the space-time density autocorrelation function of the illuminated plasma, and that under certain common and justifiable assumptions it is merely proportional to the spatial Fourier transform of this function. This close and simple relationship reduces the statistical problem to that of finding the space-time density autocorrelation function. The rest of the paper is concerned with this problem. It should be emphasized that the density autocorrelation shows up not only in this particular problem but in most fluctuation problems, so the rest of the paper should be considered, in this regard, more general in scope.

(*) Throughout this work the reader will notice a preference for the use of autocorrelation functions instead of power spectrums. There are several advantages and reasons for doing so. Although they are directly related to each other as a Fourier transform pair (Wiener theorem), this is true only in the case of stationary random process (homogeneous in the case of spatial fluctuations); whereas the concept of autocorrelation is still valid even in the case of non-stationary (and/or non-homogeneous) random processes. In most cases it is the autocorrelation function which is obtained experimentally, and therefore, it is convenient to have theoretical predictions in this form. It is also usually easier to give a physical interpretation to autocorrelation functions than it is to power spectrums. We shall see that in this particular case it is also faster (in terms of computer time) to obtain autocorrelation functions directly, by a method developed by the author and briefly described in this thesis.

The second step consists in deriving the mathematical equations that the space-time density autocorrelation function should satisfy. That is, the mathematical formulation of the problem from the physical properties of the medium. Each of the different authors who have treated this problem have taken at this stage a different approach; we take here one more different approach. This, we feel, is another important contribution to the incoherent scatter problem or more generally to the plasma fluctuation problem. All of the approaches presented so far, at this point, have their advantages and limitations which will not be discussed here. To this author's judgment the most rigorous and systematic approach is the one offered by Rostoker and Rosenbluth,¹⁸ but, the effect of "Coulomb collisions" are not included. They also need to introduce some two-time reduced distribution functions W_{11} and W_{12} and derive the dynamic equations that they are to satisfy. In Chapter III we present and prove a theorem which relates the space-time density autocorrelation function to the solution of an initial value problem of a one particle distribution function $f(\underline{x}, \underline{v}, t)$. In its more general form it is valid even for non-stationary non-homogeneous plasmas. Besides its rigor and generality, the method has the advantage of avoiding the introduction of new distribution functions W_{11}, W_{12} , and of working with the one particle distribution function and its kinetic equation which has been studied with considerably more detail and has a behavior with which one has considerably more familiarity. It turns out to be particularly convenient in the case of plasmas in thermodynamic equilibrium.

With the help of this theorem the actual mathematical problem is formulated

in Chapter IV. Necessary assumptions and approximations are taken at this stage.

From Chapter IV on, we drop some of the rigor and generality of Chapters II and III. We restrict ourselves to the thermodynamic equilibrium case. For the particular application concerning the effect of the Coulomb collisions on the ion gyro-resonances in ionospheric plasmas (which motivated this thesis), the equilibrium case is the only one worth considering. At the magnetic equator, where these resonances can be conveniently observed,^(*) thermodynamic equilibrium normally prevail at heights above 400 km during the day and at all heights during the night. At heights below 400 km it is possible to have different electron and ion temperatures. But, in such a region, gyro-resonances cannot be observed, either because of ion-neutral collisions⁵ or ion-ion Coulomb collisions.^(**) In terms of the autocorrelation this implies that the autocorrelation peak at a gyro-period does not exist. The autocorrelation function is non-zero only at relatively short times. In such a case a collisionless model is sufficient. The collisionless case with different electron and ion temperature, have already been investigated by Fejer,⁹ Salpeter,¹⁰ Rosenbluth and Rostoker,¹⁸ Buneman,¹⁴ and more fully (in-

(*) Theory shows that the ion-gyro-resonances are observed only when the illuminating wave is within a few degrees from the perpendicular to the magnetic field. The large antenna sizes required and economical considerations forces them to be installed horizontally and consequently at the magnetic equator. This is the reason for selecting the location of the Jicamarca Radio Observatory.

(**) This we claim based on our computations presented in Chapter VI for equal temperatures, qualitatively generalized to the case of different temperatures.

cluding magnetic field and numerical results) by Farley⁷ (1966).

In order to get anywhere analytically with the inclusion of a collision term one has to use a simplified collision model in the dynamical equations for f . We use a simple model of the Fokker-Planck type which could be considered as an approximation of the more complete model proposed by Dougherty,⁶ valid in the case of wavelengths smaller than the mean free path.

Dougherty proposes a model of the form:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \nu \frac{\partial}{\partial \underline{v}} (\underline{v} - \bar{\underline{v}}) f + \nu \frac{KT}{m} \frac{\partial^2}{\partial \underline{v} \cdot \partial \underline{v}} f, \quad 1.1.1$$

where $(\partial f / \partial t)_{\text{coll}}$ is the collision term, $f \equiv f(\underline{x}, \underline{v}, t)$ is the one particle distribution function, $\bar{\underline{v}}$ and T are the mean velocity and temperature at \underline{x} and t , defined by:

$$\bar{\underline{v}} = \frac{1}{n} \int \underline{v} f d\underline{v}, \quad 1.1.2$$

$$\frac{3KT}{m} = \frac{1}{n} \int (\underline{v} - \bar{\underline{v}})^2 f d\underline{v}. \quad 1.1.3$$

n is the local density defined by

$$n = \int f d\underline{v}, \quad 1.1.4$$

K is the Boltzmann constant, m the mass of the particle and ν is an inverse relaxation time which we shall refer to as the collision frequency. It is easy to

show that with the terms of the equation so defined, the number density, momentum and energy at any one point of physical space are conserved and

$(\partial f / \partial t)_{\text{coll}}$ becomes zero only when f is a maxwellian.

If we consider a small perturbation f_1 from a uniform maxwellian f_0 with density n_0 , and temperature T_0 , and zero drift velocity, equation 1.1.1 reduces to first-order to:

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = v \left(\frac{\partial}{\partial \underline{v}} \underline{v} f_1 + \frac{KT_0}{m} \frac{\partial^2}{\partial \underline{v} \cdot \partial \underline{v}} f_1 \right) + v \left(-\bar{\underline{v}} \frac{\partial f_0}{\partial \underline{v}} + \frac{KT_1}{m} \frac{\partial^2 f_0}{\partial \underline{v} \cdot \partial \underline{v}} \right), \quad 1.1.5$$

where

$$\frac{3KT_1}{m} = \frac{1}{n_0} \int v^2 f_1 d\underline{v} - \frac{n_1}{n_0} \frac{3KT_0}{m} \quad 1.1.6$$

and

$$n_1 = \int f_1 d\underline{v}. \quad 1.1.7$$

We can also write $\bar{\underline{v}}$ as $\bar{\underline{v}} = 1/n_0 \int \underline{v} f_1 d\underline{v}$.

In the case of ionospheric plasmas we have already mentioned that the collision frequency is small, both with respect to the gyro-frequency and thermal characteristic frequency $ku = 2\pi u/\lambda$ (for wavelengths λ smaller than the mean free path $\ell = u/v$). Under such conditions one can neglect the second parenthesis in 1.1.5. Although the first parenthesis in 1.1.5 is also multiplied by the same small v , it cannot be neglected. The best argument for this can be obtained from the discussion of the solution of the problem itself including such a term.

We have used a collision model which is essentially the first parenthesis in 1.1.5 generalized to multicomponent plasmas. Rough estimates of the magnitude of the neglected terms are given in Chapter IV.

In order to complete the mathematical formulation of the problem we need the values of ν as a function of the medium. In Appendix I we obtain them for the general case of multicomponent medium, starting from the Rosenbluth, MacDonald and Judd³² equations.

In Chapter V we present the solution to the mathematical problem. No further approximations are introduced. It is essentially the solution of an initial value problem for a multicomponent plasma in a magnetic field, collisions included.

The last step in the solution of the problem is the numerical evaluation of the mathematical solution. We also have something new to contribute at this stage. In Chapter VI we present a brief description of a computer technique used to obtain numerical results for the autocorrelation expression derived in Chapter V. All authors so far give their results as an expression for the power spectrum. This is given as the ratio of algebraic expressions consisting of sum and products of complex transcendental functions in integral representation. Each of the functions has to be evaluated by integrating numerically its integral representation. The autocorrelation function is then obtained by taking the Fourier transform of the power spectrum which also has to be performed numerically. Because of the magnetic field the autocorrelation function has under certain conditions an almost

periodic behavior. This behavior reflects as a very spiky frequency spectrum which degenerates to a series of delta functions when the wave vector becomes perpendicular to the magnetic field and demands relatively long computing times, becoming prohibitively long when the wave vector gets close to perpendicular. The technique presented in Chapter VI gets around most of these difficulties and limitations, obtaining the autocorrelation function directly without going through the power spectrum. It is much faster, and angles close to perpendicular present no difficulty. We have used this technique to evaluate a few autocorrelation functions showing the effect of Coulomb collisions, the dependence on the angle between wave vector and magnetic field (including very small values) and the effect of different ion composition. We have used typical values encountered in the ionosphere and an E.M. wavelength corresponding to 50 Mc as used by the Jicamarca Observatory.

2. Definition of Some Concepts and Symbols.

In order to prove with some rigor, the equivalence theorem presented in Chapter III, we need some formal definitions of the concepts used, as well as some of their properties.

Consider a system composed of s different types of particles, these being N_μ particles of type μ ($\mu = 1, 2, \dots, s$). Let \tilde{X}^μ denote a set of N_μ position vectors \tilde{x}_j^μ and velocity vectors \tilde{v}_j^μ ($j = 1, 2, \dots, N_\mu$), i.e.,

$$\underline{\underline{X}}^\mu \equiv \left(\underline{\underline{x}}_1^\mu, \underline{\underline{v}}_1^\mu; \dots; \underline{\underline{x}}_{N_\mu}^\mu, \underline{\underline{v}}_{N_\mu}^\mu \right), \quad (\mu = 1, 2, \dots, s)$$

and let

$$\underline{\underline{X}} \equiv \left(\underline{\underline{X}}^1, \underline{\underline{X}}^2, \dots, \underline{\underline{X}}^s \right) .$$

Then $\underline{\underline{X}}$ represents a point in the $6N$ -dimensional phase space for the system, where $N = N_1 + N_2 + \dots + N_s$. The exact microscopic state of the system at time t would be represented by a point in this phase-space.

In kinetic theory, the state of the system is described statistically by means of a probability distribution function $F(\underline{\underline{X}}, t)$. Here, $F(\underline{\underline{X}}, t) d\underline{\underline{X}}$ is defined to be the probability at time t of finding the representative point of the system in $d\underline{\underline{X}}$ about $\underline{\underline{X}}$, i.e., the j^{th} particle of type μ in $d\underline{\underline{x}}_j^\mu$ about $\underline{\underline{x}}_j^\mu$ with velocity in $d\underline{\underline{v}}_j^\mu$ about $\underline{\underline{v}}_j^\mu$ ($j = 1, \dots, N_\mu$; $\mu = 1, \dots, s$). From its definition F is non-negative and is normalized to unity,

$$\int F(\underline{\underline{X}}, t) d\underline{\underline{X}} = 1 .$$

Moreover, F is required to be symmetric with respect to interchange of the labels of like particles. The equation of motion for F is the Liouville equation

$$\frac{\partial F}{\partial t} + \{H, F\} = 0 , \quad 1.2.0$$

where H is the Hamiltonian of the system of N particles and $\{H, F\}$ denotes the Poisson bracket.

The value of any macroscopic quantity at time t is determined as the ex-

pectation of the corresponding microscopic quantity. Thus, for example, the microscopic density $n_{\mu}(\underline{y}, t)$ of μ -particles is defined as

$$n_{\mu}(\underline{y}, t) = \sum_{j=1}^N \delta(\underline{x}_j^{\mu} - \underline{y}) . \quad 1.2.1$$

The corresponding macroscopic density $\langle n_{\mu}(\underline{y}, t) \rangle$ is then given by:

$$\langle n_{\mu}(\underline{y}, t) \rangle = \int n_{\mu}(\underline{y}, t) F(\underline{X}, t) d\underline{X} = \int d\underline{X} F(\underline{X}, t) \sum_{j=1}^N \delta(\underline{x}_j^{\mu} - \underline{y}) . \quad 1.2.2$$

We shall also be interested in macroscopic variables that refer to two positions and two times. Such macroscopic variables are conveniently expressed in terms of the joint distribution function, $D(\underline{X}, t; \underline{X}', t')$. Here, $D(\underline{X}, t; \underline{X}', t')$ $d\underline{X} d\underline{X}'$ is the joint probability that the system will be found at time t' in $d\underline{X}'$ about \underline{X}' , and at time t in $d\underline{X}$ about \underline{X} . It is also advantageous to introduce the conditional probability, $F_G(\underline{X}, t | \underline{X}', t')$ with the interpretation that $F_G(\underline{X}, t | \underline{X}', t')$ $d\underline{X}$ is the probability at time t of finding the system in $d\underline{X}$ about \underline{X} given that it was certainly at \underline{X}' at time t' . It then follows that

$$D(\underline{X}, t; \underline{X}', t') = F(\underline{X}', t') F_G(\underline{X}, t | \underline{X}', t') . \quad 1.2.3$$

It should be noted that the conditional probability F_G is a singular function involving products of delta functions; if the exact microscopic state \underline{X}' is given at time t' the Liouville equation determines the exact microscopic state \underline{X} at time t .

An important concept in regard to the incoherent scattering problem, and

in general with the problem of fluctuations in a plasma, is the space-time density autocorrelation function $\langle n_\mu(\underline{y}, t) n_\mu(\underline{y}', t') \rangle$. This is defined as:

$$\langle n_\mu(\underline{y}, t) n_\mu(\underline{y}', t') \rangle = \int d\underline{X} d\underline{X}' \sum_{i=1}^N \sum_{j=1}^N \delta(\underline{x}_i^\mu - \underline{y}) \delta(\underline{x}_j^\mu - \underline{y}') D(\underline{X}, \underline{X}', t, t'). \quad 1.2.4$$

We shall also use the symbol $\rho_\mu(\underline{r}, \tau; \underline{y}, t) \equiv \langle n_\mu(\underline{y}, t) n_\mu(\underline{y}', t') \rangle$, where $\underline{r} = \underline{y} - \underline{y}'$ and $\tau = t - t'$.

Normally (and fortunately) one can express most macroscopical quantities in terms of much simpler reduced distribution functions, mainly the one-particle distribution function $f^\mu(\underline{x}, \underline{v}, t)$ and the two-particle distribution functions:

$f_2^{\mu\eta}(\underline{x}_1, \underline{x}_2, \underline{v}_1, \underline{v}_2, t)$ and $f_2^{\mu\mu}(\underline{x}_1, \underline{x}_2, \underline{v}_1, \underline{v}_2, t)$. These are defined as:

$$f^\mu(\underline{x}_1^\mu, \underline{v}_1^\mu, t) = N \int_{(\mu_1)} d' \underline{X} F(\underline{X}, t), \quad 1.2.5$$

$$f_2^{\mu\eta}(\underline{x}_1^\mu, \underline{x}_2^\eta, \underline{v}_1^\mu, \underline{v}_2^\eta, t) = N_\mu N_\eta \int_{(\mu_1, \eta_2)} d'' \underline{X} F(\underline{X}, t), \quad 1.2.6$$

and

$$f_2^{\mu\mu}(\underline{x}_1^\mu, \underline{x}_2^\mu, \underline{v}_1^\mu, \underline{v}_2^\mu, t) = N_\mu (N_\mu - 1) \int_{(\mu_1, \mu_2)} d'' \underline{X} F(\underline{X}, t). \quad 1.2.7$$

Here, $\int_{(\mu_j)} d' \underline{X}$ and $\int_{(\mu_1, \eta_j)} d'' \underline{X}$ denote symbolically integration over all variables other than $\underline{x}_j^\mu, \underline{v}_j^\mu$ and $\underline{x}_1^\mu, \underline{v}_1^\mu, \underline{x}_j^\eta, \underline{v}_j^\eta$ respectively. Which particles i and j are singled out is immaterial since $F_N(\underline{X}, t)$ is invariant under interchange of particles of the same specie. We shall take in general particles 1 and 2.

Using equation 1.2.5 in 1.2.2 and the invariance of $F_N(\underline{X}, t)$ under particle

interchange one can write the density $\langle n_\mu(\underline{y}, t) \rangle$ in its more common forms:

$$\langle n_\mu(\underline{y}, t) \rangle = \int d\underline{x} d\underline{v} \delta(\underline{x} - \underline{y}) f^\mu(\underline{x}, \underline{v}, t) = \int d\underline{v} f^\mu(\underline{y}, \underline{v}, t) . \quad 1.2.8$$

Similarly time autocorrelation functions can be expressed in terms of two-time reduced distribution functions $W_{11}^\mu, W_{12}^{\mu\mu}, W_{12}^{\mu\eta}$ defined by:

$$W_{11}^\mu(\underline{x}_1^\mu, \underline{x}'_1^\mu, \underline{v}_1^\mu, \underline{v}'_1^\mu, t, t') = N_\mu \int_{(\mu_1)} d'X d'X' D(\underline{X}, t, \underline{X}', t') , \quad 1.2.9$$

$$W_{12}^{\mu\mu}(\underline{x}_1^\mu, \underline{x}_2^\mu, \underline{x}'_2^\mu, \underline{v}_1^\mu, \underline{v}'_2^\mu, t, t') = N_\mu (N_\mu - 1) \int_{(\mu_1) (\mu_2)} d'X d'X' D(\underline{X}, t, \underline{X}', t') , \quad 1.2.10$$

$$W_{12}^{\mu\eta}(\underline{x}_1^\mu, \underline{x}_2^\eta, \underline{v}_1^\mu, \underline{v}'_2^\eta, t, t') = N_\mu N_\eta \int_{(\mu_1) (\eta_2)} d'X d'X' D(\underline{X}, t, \underline{X}', t') . \quad 1.2.11$$

Where as before the prime in d' stands for the omission from integration of the particle in the parenthesis underneath. In terms of these functions one can express $\langle n_\mu(\underline{y}, t) n_\mu(\underline{y}', t') \rangle$ as:

$$\begin{aligned} \langle n_\mu(\underline{y}, t) n_\mu(\underline{y}', t') \rangle &= \int d\underline{x} d\underline{x}' d\underline{v} d\underline{v}' \delta(\underline{x} - \underline{y}) \delta(\underline{x}' - \underline{y}') W_{12}^{\mu\mu}(\underline{x}, \underline{x}', \underline{v}, \underline{v}', t, t') \\ &+ \int d\underline{x} d\underline{x}' d\underline{v} d\underline{v}' \delta(\underline{x} - \underline{y}) \delta(\underline{x}' - \underline{y}') W_{11}^{\mu\mu}(\underline{x}, \underline{x}', \underline{v}, \underline{v}', t, t') . \end{aligned} \quad 1.2.12$$

It is these functions $W_{11}^{\mu\mu}, W_{12}^{\mu\mu}, W_{12}^{\mu\eta}$ and their corresponding dynamic equations that we referred to before and that we avoid by means of the equivalence theorem presented in Chapter III.

We say a system is time-stationary whenever its statistical properties (mainly F and D) are invariant under a time displacement, i.e.:

$$F(\underline{X}, t) = F(\underline{X}, t + t_0) \quad \text{and} \quad D(\underline{X}, t, \underline{X}', t') = D(\underline{X}, t + t_0, \underline{X}', t' + t_0)$$

for any t_0 . This implies that we can write F as well as any reduced distribution or macroscopic quantity derived from it as independent function of time, and D as well as any two-time reduced distribution or macroscopic quantity (space-time correlations) as functions of their time difference $\tau = t - t'$ only. In practice we say a system is time-stationary in a less strict sense, when the system is practically stationary within the time scale of interest. In this sense the system may not be in thermodynamical equilibrium.

We say a system is spatially homogeneous whenever its statistical properties, given mainly by F and D are invariant under a coordinate displacement in physical space, i.e., when $F(\underline{X}, t) = F(\underline{X} + \{\underline{a}\}, t)$ and $D(\underline{X}, t, \underline{X}', t') = D(\underline{X} + \{\underline{a}\}, t, \underline{X}' + \{\underline{a}\}, t')$, where $\{\underline{a}\}$ is any $6 \times N$ dimensional vector with components $\underline{x}_i^\mu = \underline{a}$ and $\underline{y}_i^\mu = 0$ for all i 's and μ 's. In a strict sense only a system that is infinite in ^(time) extent can be ^(time-stationary) spatially homogeneous. Dealing with such a system usually brings some difficulties with the normalization and definition of the reduced distribution functions. This difficulty can be avoided if we consider the system as the limit of a locally homogeneous system confined in a volume V , in which N_1, N_2, \dots, N_s as well as V go to infinity in such a way as to keep $n_\mu = N_\mu / V$ constant for all μ 's. In practice a system is considered locally homogeneous when

it is practically invariant under space displacements within a region of interest. In this sense the system can be finite in extent (which is usually the case). The space invariance condition implies that all reduced distribution functions depend on the physical space coordinates \tilde{x}_i^μ and \tilde{x}_j^η through their differences $\tilde{x}_i^\mu - \tilde{x}_j^\eta$ or $\tilde{x}_i^\mu - \tilde{x}_j'^\eta$, and that the one particle distribution function f^μ as well as any macroscopic observable related to it are independent of \tilde{x} . This corresponds to the intuitional concept of homogeneity.

CHAPTER II

RELATIONSHIP BETWEEN THE AUTOCORRELATION OF THE SCATTERED WAVE AND $\langle n_e(\underline{y}, t) n_e(\underline{y}', t') \rangle$

Let us consider a general configuration as the one shown in Figure 1. A transmitting antenna A illuminates a plasma with an electromagnetic wave of frequency ω_0 and a receiving antenna B "looks" at some region of the illuminated plasma. We shall find a functional relationship between the statistical properties of the signals received at B and the electron space-time density correlation function of the plasma. For the particular ionospheric applications of interest, it is possible to assume that the plasma is homogeneous and stationary and that the fields of both antennas are plane waves in the common region. For the sake of generality we shall not initially assume homogeneity and stationarity. This latter case will appear as a particular case of the more general situation.

The electromagnetic fields at any given point due to the presence of the illuminating antenna can be represented as the sum of two fields: a coherent field and an incoherent field. By coherent field we mean that part of the field whose frequency is equal to the transmitter frequency, and a phase which depends on the macroscopic properties of the medium. The determination of such a field is a wave propagation problem. It implies the knowledge of the dielectric properties

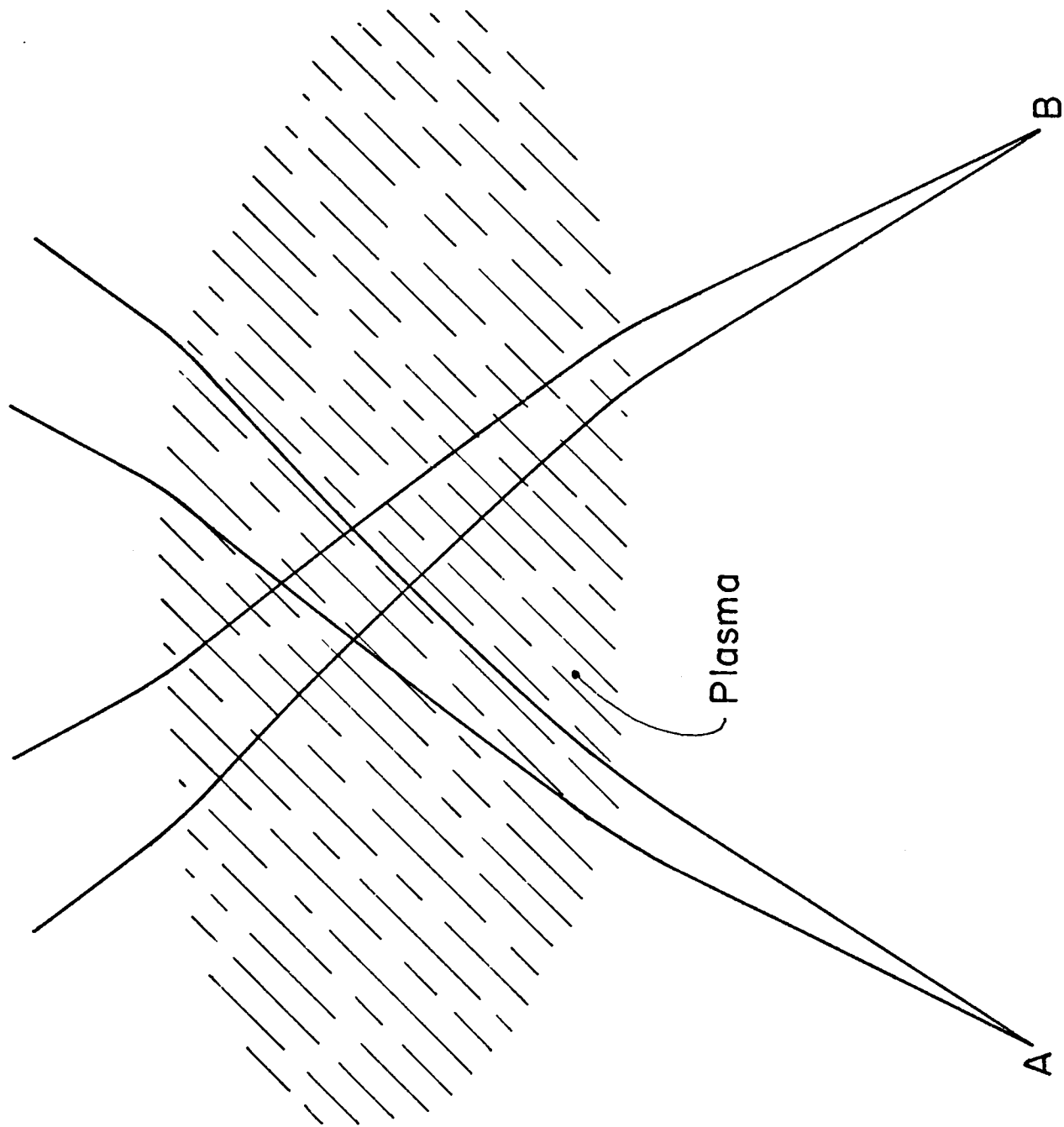


FIG. 1. General incoherent scattering configuration

of the medium and we shall assume these to be known. By incoherent field we mean that part of the field due to scattering of the coherent field by the individual electrons in a direction other than the direction of the coherent field. This field is stochastic in nature due to thermal motion of the electrons and has a frequency spectrum with some finite width. The frequency spread is due to the Doppler shift imposed on the scattered signal by the moving electrons. Strictly speaking there is a contribution to the incoherent field due to the re-scattering by the electrons of the already incoherent field. We shall assume that such "secondary scattering" has a negligible effect. We exclude from our treatment scattering in the direction of the coherent field, since in such a case the scattered signal is of the same frequency as the incident field and forms part of the coherent field. The evaluation of the dielectric properties of a medium is in fact equivalent to determination of the effects of the scattering (by the electrons) of the field in this direction. To obtain these properties, one takes a completely different approach. We are not concerned with that problem here. We also neglect the ion contribution to the scattered field.

Under these assumptions we can write that the incoherent electric field $\tilde{E}_B(t)$ at B is due to the superposition of the scattered fields due to each one of the electrons. To this field there corresponds a scalar field $E(t)$ across the terminals of the receiving antenna; this scalar field can be written as the real part of a complex quantity:

$$E(t) = \text{Re} \sum_i^{N_e} \chi(\underline{x}_i^e, t) e^{-i\omega_0 t + \varphi_T} . \quad 2.1.1.$$

Here $\chi(\underline{y}, t)$ is some function which linearly relates the signal at the receiver terminals, due to the presence of one electron at \underline{y} , with the driving field $E_T e^{-i\omega_0 t + \varphi_T}$ somewhere across the transmitter terminals. In writing equation 2.1.1 we implicitly assume that the electron can be considered as a scattering center, i.e., the scattering oscillation of the electron is much smaller than the wavelength. We shall use the physically real signal $E(t)$ since some of the following operations with $E(t)$ are non-linear and the complex notation loses its convenience. We show explicitly the time dependence of χ to take care of any possible macroscopic changes of the medium. In order to be able to talk about coherent fields we have to limit such time dependence to very slow variations compared to the signal period ω_0^{-1} . We have introduced for convenience a random phase φ_T with probability distribution $P(\varphi_T) = 1/2\pi$ for $0 < \varphi_T < 2\pi$, the reasons will become apparent soon and physically corresponds to our ignorance of the actual phase of transmitter signal.

We should emphasize that writing $E(t)$ as a superposition of the contributions of each of the electrons does not imply that their contributions are statistically independent (Thomson-Rayleigh scattering). The positions of the different electrons are in fact strongly correlated to each other, and to the positions of the ions. These strong correlations will be taken into account by averaging with the proper probability distribution function which at no time assumes such independence.

The field $E(t)$ is a stochastic function of time, owing to the random nature of the location of the electrons $\underline{x}_i^e \equiv \underline{x}_i^e(t)$. Statistically we can characterize it by its autocorrelation function $\langle E(t) E(t+\tau) \rangle$ given by:

$$\begin{aligned} \langle E(t) E(t+\tau) \rangle = & \int d\underline{X} d\underline{X}' \int d\varphi_T \left[\mathcal{R}e \sum_{i=1}^{N_e} \chi(\underline{x}_i^e) e^{-i(\omega_0 t + \varphi_T)} \right] \\ & \times \left[\mathcal{R}e \sum_{j=1}^{N_e} \chi'(\underline{x}_j^e) e^{-i(\omega_0 t + \omega_0 \tau + \varphi_T)} \right] \times D(\underline{X}', t+\tau, \underline{X}, t) P(\varphi_T) , \end{aligned} \quad 2.1.2$$

where $\chi(\underline{x}_i^e) \equiv \chi(\underline{x}_i^e; t)$ and $\chi'(\underline{x}_j^e) \equiv \chi(\underline{x}_j^e; t+\tau)$. We can write equation 2.1.2 in the form:

$$\begin{aligned} \langle E(t) E(t+\tau) \rangle = & \frac{1}{2} \mathcal{R}e e^{i\omega_0 \tau} \int d\underline{X} d\underline{X}' \sum_{i=1}^{N_e} \sum_{j=1}^{N_e} |\chi(\underline{x}_i^e)| |\chi'(\underline{x}_j^e)| \\ & \times \exp \left[\varphi(\underline{x}_i^e) - \varphi'(\underline{x}_j^e) \right] D(\underline{X}', t+\tau, \underline{X}, t) \\ & + \frac{1}{2} \mathcal{R}e e^{-i(\omega_0 \tau + 2\omega_0 t)} \frac{1}{2\pi} \int d\varphi_T e^{-i2\varphi_T} \int d\underline{X} d\underline{X}' |\chi(\underline{x}_i^e)| |\chi'(\underline{x}_j^e)| \\ & \times \exp \left[\varphi(\underline{x}_i^e) + \varphi'(\underline{x}_j^e) \right] D(\underline{X}', t+\tau, \underline{X}, t) , \end{aligned} \quad 2.1.3$$

where we have written $\chi(\underline{x}_i^e) = |\chi(\underline{x}_i^e)| \exp [\varphi(\underline{x}_i^e)]$. The second integral does not contribute since $\int_0^{2\pi} e^{-i2\varphi_T} d\varphi_T = 0$. The introduction of a random phase is a formal way of getting rid of the second integral. In practice averages are taken with respect to time, in which case the second integral would also go to zero

because of the $e^{-i2\omega_0 t}$ factor. We can write equation 2.1.3 in the form:

$$\begin{aligned} \langle E(t) E(t+\tau) \rangle &= \frac{1}{2} \Re e \, e^{i\omega_0 \tau} \int d\mathbf{y} \, d\mathbf{y}' \, \chi(\mathbf{y}) \, \chi'^*(\mathbf{y}') \\ &\quad \times \int d\mathbf{X} \, d\mathbf{X}' \sum_{i=1}^{N_e} \sum_{j=1}^{N_e} \delta(\mathbf{x}_i^e - \mathbf{y}) \, \delta(\mathbf{x}_j'^e - \mathbf{y}') \, D(\mathbf{X}', t+\tau, \mathbf{X}, t), \end{aligned} \quad 2.1.4$$

and using our definition for $\langle n_e(\mathbf{y}, t) n_e(\mathbf{y}', t') \rangle$

$$\langle E(t) E(t+\tau) \rangle = \frac{1}{2} \Re e \, e^{i\omega_0 \tau} \int d\mathbf{y} \, d\mathbf{y}' \, \chi(\mathbf{y}) \, \chi'^*(\mathbf{y}') \langle n_e(\mathbf{y}, t) n_e(\mathbf{y}', t+\tau) \rangle. \quad 2.1.5$$

This is a very general relationship. It is valid for general antenna field configurations and for non-homogeneous non-stationary plasmas. This, of course, within the limitations implied by our original assumptions. Expression 2.1.5 separates the problem into two distinct ones. The first is a wave propagation problem (in a plasma) to determine the function $\chi(\mathbf{y})$; that is the determination of the signal at the receiving antenna terminal due to the scattering of radiation by a single free electron located at \mathbf{y} and subject to whatever coherent field the transmitting antenna produces at that location. The second is a statistical physics problem to determine the density autocorrelation function $\langle n_e(\mathbf{y}, t) n_e(\mathbf{y}', t') \rangle$. We are mainly interested in the second problem.

Let us evaluate $\chi(\mathbf{y})$ in the case, to which most practical problems are reduced, of a homogeneous plasma in which the antennas have been removed sufficiently far so that their fields can be approximated as plane waves. The operating frequency, of course, would have to be higher than the plasma frequency

for the wave to exist. To this approximation the amplitude $|\chi(\underline{y})|$ is a constant and its phase $\varphi(\underline{y})$ is given by $\varphi(\underline{y}) = \varphi_0 + \underline{k}_i \cdot \underline{y} - \underline{k}_r \cdot \underline{y}$, where \underline{k}_i and \underline{k}_r are wave vectors in direction from and to the transmitting and receiving antenna respectively and magnitude ω_0/c_p and c_p is the phase velocity in the medium. $\chi(\underline{y})$ is in this case then given by

$$\chi(\underline{y}) = A e^{-i\mathbf{k} \cdot \underline{y}}, \quad 2.1.6$$

where $\mathbf{k} = \underline{k}_i - \underline{k}_r$ and A is a constant $A = |A| e^{i\varphi_0}$. We will not be concerned* with the evaluation of A . The only important fact for us is that it is constant. It may be worthwhile to mention that it is proportional to the classical electron radius $r_e = e^2/m c^2$, and $\sin \theta$, the angle between the direction of \underline{k}_r and the direction of the electrical field $\underline{E}(\underline{y})$ at \underline{y} (see Thomson scattering, Jackson²³). Placing equation 2.1.6 into 2.1.5, we obtain the more usual expression:

$$\langle E(t) E(t+\tau) \rangle = \mathcal{R}e \frac{1}{2} |A|^2 e^{i\omega_0\tau} \int d\underline{y} d\underline{y}' e^{i\mathbf{k}(\underline{y}-\underline{y}')} \langle n_e(\underline{y},t) n_e(\underline{y}',t+\tau) \rangle, \quad 2.1.7$$

and since in homogeneous systems $\langle n_e(\underline{y},t) n_e(\underline{y}',t+\tau) \rangle = \rho_e(\underline{r},\tau;t)$, where $\underline{r} = \underline{y} - \underline{y}'$

$$\langle E(t) E(t+\tau) \rangle = \mathcal{R}e \frac{1}{2} |A|^2 e^{i\omega_0\tau} \int d\underline{y}' \int d\underline{r} e^{-i\mathbf{k} \cdot \underline{r}} \rho_e(\underline{r},\tau;t). \quad 2.1.8$$

The non-convergent integral $\int d\underline{y}'$ is due to our assumption of an (infinite) homogeneous plasma and (infinite) plane waves. Physically it makes sense, since such

*

For a derivation of the absolute value of received Power, see K. Bowles, Gr. Ochs, and J. L. Green.²²

an infinite system would give us an infinite amount of scattered power. In any case, the usefulness of equation 2.1.8 is in the integrand which is finite, and can be interpreted as a "per unit volume" contribution to $\langle E(t) E(t+\tau) \rangle$.

In practice a finite volume, V , is defined by the weighting function of the intersecting beam patterns or, in the case of transmission and reception from the same location, by time delay discrimination in a radar fashion. Assuming an even weighting factor throughout the volume one can then write:

$$\langle E(t) E(t+\tau) \rangle = \frac{1}{2} \text{Re} |A|^2 e^{i\omega_0 \tau} V \int d\tilde{r} e^{-i\tilde{k} \cdot \tilde{r}} \rho_c(\tilde{r}, \tau; t) .$$

We notice that the integral expression is the spatial Fourier transform of

$\rho_c(\tilde{r}, \tau; t)$:

$$\bar{\rho}_c(\tilde{\xi}, \tau; t) = \int d\tilde{r} e^{-i\tilde{\xi} \cdot \tilde{r}} \rho_c(\tilde{r}, \tau; t) , \quad 2.1.9$$

evaluated at $\tilde{\xi} = \tilde{k}$. Therefore, we can write

$$\langle E(t) E(t+\tau) \rangle = \frac{1}{2} \text{Re} |A|^2 e^{i\omega_0 \tau} V \bar{\rho}_c(\tilde{k}, \tau; t) \Big|_{\tilde{\xi} = \tilde{k}} . \quad 2.1.10$$

CHAPTER III

ON THE DYNAMICS OF $\langle n_{\sigma}(\underline{y},t) n_{\sigma}(\underline{y}',t') \rangle$

— AN EQUIVALENCE THEOREM*

We have just shown that the statistical part of the incoherent scattering problem is reduced to the evaluation of $\langle n_e(\underline{y},t) n_e(\underline{y}',t') \rangle$. Here, we present a theorem which enables us to solve such a problem in terms of a more familiar one, mainly that of finding the time evolution of the familiar one-particle distribution function f from a specified initial value. The statement of the theorem and its proof are exact, and apply in the most general conditions; necessary approximations come about only in trying to solve the exactly formulated problem. Although we shall use this theorem in the following chapters only for relatively simple conditions, mainly homogeneous stationary plasmas, we present it in its most general form; since its proof, with the exception of more cumbersome notation, is essentially the same.

Let $F = F(\underline{X},t)$ be the N -body distribution function which describes our system. It has been specified at some initial time $t=0$ so that:

* After I conceived this idea, J. Weinstock brought to my attention a paper of his in which he derives essentially the same property of $\langle n_{\sigma}(\underline{y},t) n_{\sigma}(\underline{y}',t') \rangle$. For a different approach see reference 25.

$$F(\underline{X}, 0) = P(\underline{X}) \quad . \quad 3.1.1$$

At any other arbitrary time t , $F(\underline{X}, t)$ is given as the solution of Liouville's equation,

$$\frac{\partial F}{\partial t} + \{H, F\} = 0 \quad . \quad 3.1.2$$

Let us construct a hypothetical system with N -body distribution function $F' \equiv F'(\underline{X}', \tau)$. It is to evolve in time τ in a $6N$ -dimensional space \underline{X}' according to a Liouville equation,

$$\frac{\partial F'}{\partial \tau} + \{H, F'\} = 0 \quad , \quad 3.1.3$$

with the same Hamiltonian as the actual system. We specify this system by its initial condition at time $\tau=0$:

$$F'(\underline{X}', 0) = \frac{\sum_{\sigma}^N \delta(\underline{x}'^{\sigma} - \underline{y}) F(\underline{X}, t)}{\langle n_{\sigma}(\underline{y}, t) \rangle_F} \quad , \quad 3.1.4$$

where

$$\langle n_{\sigma}(\underline{y}, t) \rangle_F = \int d\underline{X} \sum_{\sigma}^N \delta(\underline{x}_i^{\sigma} - \underline{y}) F(\underline{X}, t) \quad 3.1.5$$

is the macroscopic density of the actual system. We note that to each point \underline{y} in physical space and to each time t there corresponds a different $F'(\underline{X}, 0)$. We shall display this parametric dependance explicitly by writing:*

* Arguments after a semicolon are to be taken as parameters.

$$F'(\underline{X}', \tau; \underline{y}, t) \equiv F(\underline{X}', \tau) . \quad 3.1.6$$

It should also be noted that $F'(\underline{X}, 0)$ is non-negative, normalized to one, and invariant under particle interchange of the same species. Therefore, it conforms with the requirements for it to be a physically acceptable distribution function.

At any point $\underline{y}' = \underline{y} + \underline{r}$ in physical space and time $t' = t + \tau$ the macroscopic density of the hypothetical system $\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle$ is given by definition as

$$\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle_{F'} = \int d\underline{X}' \sum_j^N \delta(\underline{x}_j^{\sigma} - (\underline{y} + \underline{r})) F'(\underline{X}', \tau; \underline{y}, t) . \quad 3.1.7$$

To avoid confusion we shall write F and F' by the expectation bracket to indicate with respect to which distribution function they are defined. We shall also prime those functions which refer to the hypothetical system. We claim the following:

THEOREM: — "The space-time density autocorrelation function of species σ :

$$\rho_{\sigma}(\underline{r}, \tau; \underline{y}, t) \equiv \langle n_{\sigma}(\underline{y}, t) n_{\sigma}(\underline{y} + \underline{r}, t + \tau) \rangle$$

in a multicomponent plasma (or statistical system) with N -body probability distribution function $F(\underline{X}, t)$ can be written, for $\tau > 0$, as the product of two densities:

$$\rho_{\sigma}(\underline{r}, \tau; \underline{y}, t) = \langle n_{\sigma}(\underline{y}, t) \rangle_F \langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle_{F'} ; \quad 3.1.8$$

the density $\langle n_{\sigma}(\underline{y}, t) \rangle_F$ of species σ of the actual system, and the density $\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle_{F'}$ of the same species in a hypothetical system as defined above

(equations 3.1.3 and 3.1.4)."

The proof of the theorem is based on identifying the conditional probability $F_G(\underline{X}', t' | \underline{X}, t)$ with the Green's function of the Liouville equation 3.1.3.

The Green's function $G \equiv G(\underline{X}', \tau; \underline{X})$ in the $6N$ -dimensional space \underline{X}' is defined as the solution of

$$\frac{\partial G}{\partial \tau} + \{H, G\} = \delta(\underline{X}' - \underline{X}) \delta(\tau) \quad \text{for} \quad \tau \geq 0, \quad 3.1.9$$

where G is operated on through the \underline{X}' variable, and satisfies the "casuality condition"

$$G = 0 \quad \text{for} \quad \tau < 0. \quad 3.1.10$$

By integrating over a infinitesimal time around $\tau=0$, one can show that for $\tau > 0$, G is also a solution of the initial value problem

$$\frac{\partial G}{\partial \tau} + \{H, G\} = 0 \quad \text{for} \quad \tau > 0 \quad 3.1.11$$

$$G = \delta(\underline{X}' - \underline{X}) \quad \text{at} \quad \tau = 0. \quad 3.1.12$$

We can interpret G as the probability at time $t' = t + \tau$ for the system to be found at $d\underline{X}'$ about \underline{X}' given that it was certainly at \underline{X}' at time t ($\tau=0$). But this is the definition of the conditional probability $F_G(\underline{X}', t' | \underline{X}, t)$ (see page I-13). Thus, we can write the joint distribution function (equation 1.2.3):

$$D(\underline{X}', t + \tau, \underline{X}, t) = F(\underline{X}, t) G(\underline{X}', \tau; \underline{X}) \quad \text{for} \quad \tau > 0. \quad 3.1.13$$

We are able to write the time dependence of G through the difference $\tau = t' - t$

because of the time invariance of the Liouville operator (we assume H to be time independent).

We can formally write the solution of equation 3.1.3 with initial conditions

3.1.4 in terms of the Green's function $G(\underline{X}', \tau; \underline{X})$:*

$$\begin{aligned} F'(\underline{X}', \tau; \underline{y}, t) &= \int d\underline{X} G(\underline{X}', \tau; \underline{X}) F'(\underline{X}, 0; \underline{y}, t) \\ &= \int d\underline{X} G(\underline{X}', \tau; \underline{X}) \frac{\sum_i^{N_\sigma} \delta(\underline{x}_i^\sigma - \underline{y}) F(\underline{X}, t)}{\langle n_\sigma(\underline{y}, t) \rangle_F} . \end{aligned} \quad 3.1.14$$

Now, the space-time density autocorrelation function $\rho_\sigma(\underline{r}, \tau; \underline{y}, t)$ is by definition given by

$$\rho_\sigma(\underline{r}, \tau; \underline{y}, t) = \int d\underline{X} d\underline{X}' \sum_i^{N_\sigma} \sum_j^{N_\sigma} \delta(\underline{x}_i^\sigma - \underline{y}) \delta(\underline{x}_j^\sigma - (\underline{y} + \underline{r})) D(\underline{X}', t + \tau, \underline{X}, t). \quad 3.1.15$$

Using equation 3.1.13 and reordering terms we can write 3.1.15 above as:

$$\begin{aligned} \rho_\sigma(\underline{r}, \tau; \underline{y}, t) &= \langle n_\sigma(\underline{y}, t) \rangle_F \int d\underline{X}' \sum_j^{N_\sigma} \delta(\underline{x}_j'^\sigma - (\underline{y} + \underline{r})) \int d\underline{X} \\ &\quad \sum_i^{N_\sigma} \delta(\underline{x}_i^\sigma - \underline{y}) F(\underline{X}, t) \\ &\quad \times G(\underline{X}', \tau; \underline{X}) \frac{i}{\langle n_\sigma(\underline{y}, t) \rangle_F} \text{ for } \tau > 0, \end{aligned} \quad 3.1.16$$

where we have multiplied and divided by $\langle n_\sigma(\underline{y}, t) \rangle_F$. Here, we recognize our

* see Balescue's²⁴, § 4.

expression for $F'(\underline{X}', \tau; \underline{y}, t)$ as given by the formal solution 3.1.14 of the hypothetical initial value problem. In terms of this function we can write:

$$\rho_{\sigma}(\underline{r}, \tau; \underline{y}, t) = \left\langle n_{\sigma}(\underline{y}, t) \right\rangle_F \int d\underline{X}' \sum_j^{N_{\sigma}} \delta(\underline{x}'_j - (\underline{y} + \underline{r})) F'(\underline{X}', \tau; \underline{y}, t) \quad \text{for } \tau > 0 \quad 3.1.17$$

and in terms of the macroscopic density $\left\langle n'_{\sigma}(\underline{y} + \underline{r}, t; \underline{y}, t) \right\rangle_{F'}$, of the hypothetical system as defined in 3.1.7:

$$\rho_{\sigma}(\underline{r}, \tau; \underline{y}, t) = \left\langle n_{\sigma}(\underline{y}, t) \right\rangle_F \left\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \right\rangle_{F'} \quad \text{for } \tau > 0, \quad 3.1.18$$

which is the main statement of the theorem.

We reduce in this way the problem of finding the dynamics of $\rho_{\sigma}(\underline{r}, \tau; \underline{y}, t)$ to that of finding the densities of two initial value problems: the density of the actual system and the density of a hypothetical one "prepared" at the initial time in accordance with the initial conditions 3.1.4. In the case of time-stationary systems the problem is reduced to finding the latter density only.

Although we have written in the process of proving the theorem, a formal solution in terms of the Green's function, it is essentially impossible to find such a function. On the other hand, as stated in the introduction, densities can be expressed in terms of the much simpler one particle distribution functions, f^{η} (one for each different species). It is one of the main tasks of kinetic theory of gases and/or plasmas to obtain a dynamic equation for the time evolution of such functions. One of the main efforts is to obtain them in a closed form so that they depend only on the functions themselves and their initial conditions. That there are some assumptions and

approximations in deriving such equations does not concern us here. Whatever the form that they may take, they require (for an unambiguous solution) that the initial conditions be specified. We obtain this information from our theorem.

For the actual system, with density

$$\langle n_{\sigma}(\underline{y}, t) \rangle = \int d\underline{v} d\underline{x} \delta(\underline{x} - \underline{y}) f^{\sigma}(\underline{x}, \underline{v}, t) , \quad 3.1.19$$

the initial distribution $f^{\sigma}(\underline{x}, \underline{v}, 0)$, as well as the initial distributions $f^{\eta}(\underline{x}, \underline{v}, 0)$, $\eta \neq \sigma$, for the other species, is the least that we can ask in order to specify the system.

For the hypothetical system we obtain the conditional density $\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle_{F'}$ from:

$$\langle n'_{\sigma}(\underline{y} + \underline{r}, \tau; \underline{y}, t) \rangle_{F'} = \int d\underline{v} d\underline{x} \delta(\underline{x} - (\underline{y} + \underline{r})) f'^{\sigma}(\underline{x}, \underline{v}, \tau; \underline{y}, t) , \quad 3.1.20$$

where $f'^{\sigma}(\underline{x}, \underline{v}, \tau; \underline{y}, t)$ is the one particle distribution of species σ of the hypothetical system. This distribution satisfies a kinetic equation of exactly the same form as the one for the actual system of the same species (both systems have the same Hamiltonian). This equation is coupled to the kinetic equations of distribution functions $f'^{\eta}(\underline{x}, \underline{v}, \tau; \underline{y}, t)$ for all other species $\eta \neq \sigma$, which are also of the same form as the corresponding ones in the actual system. They differ only in their initial conditions and that they evolve in time τ rather than t .

We obtain the initial conditions for the hypothetical system by reducing the $6N$ initial distribution $F'(\underline{X}, 0; \underline{y}, t)$ as given in 3.1.4. They have two different forms

depending whether it is the one particle distribution function of the σ species, i.e., the species in whose density autocorrelation we are interested, or is the distribution of any of the other species $\eta \neq \sigma$.

For the σ species we have:

$$\begin{aligned} f'^{\sigma}(\underline{x}_1^{\sigma}, \underline{v}_1^{\sigma}, \tau; \underline{y}, t) \Big|_{\tau=0} &= N_{\sigma} \int d'X F'(X, 0; \underline{y}, t) \\ &= N_{\sigma} \int_{(\sigma_1)} d'X \frac{\sum_i^N \delta(\underline{x}_i^{\sigma} - \underline{y}) F(X, t)}{\langle n_{\sigma}(\underline{y}, t) \rangle_F}, \end{aligned} \quad 3.1.21$$

and making use of the invariance of F under particle interchange:

$$\begin{aligned} f'^{\sigma}(\underline{x}_1^{\sigma}, \underline{v}_1^{\sigma}, \tau; \underline{y}, t) \Big|_{\tau=0} &= \frac{\delta(\underline{x}_1^{\sigma} - \underline{y})}{\langle n_{\sigma}(\underline{y}, t) \rangle_F} N \int_{(\sigma_1)} d'X F(X, t) \\ &\quad + \frac{1}{\langle n_{\sigma}(\underline{y}, t) \rangle} \int d\underline{x}_1^{\sigma} d\underline{v}_1^{\sigma} \delta(\underline{x}_1^{\sigma} - \underline{y}) \\ &\quad \times N_{\sigma}(N_{\sigma} - 1) \int_{(\sigma_1, \sigma_2)} d''X F(X, t). \end{aligned} \quad 3.1.22$$

But by definition (1.2.5 and 1.2.7)

$$f^{\sigma}(\underline{x}_1^{\sigma}, \underline{v}_1^{\sigma}, t) = N_{\sigma} \int_{(\sigma_1)} d'X F(X, t) \quad 3.1.23$$

and

$$f_2^{\sigma\sigma}(\underline{x}_1^\sigma, \underline{x}_2^\sigma, \underline{v}_1^\sigma, \underline{v}_2^\sigma, t) = N_\sigma(N_\sigma - 1) \int_{(\sigma_1, \sigma_2)} d''X F(\underline{X}, t) . \quad 3.1.24$$

Therefore:

$$\begin{aligned} f'^{\sigma}(\underline{x}_1^\sigma, \underline{v}_1^\sigma, \tau; \underline{y}, t) \Big|_{\tau=0} &= \frac{\delta(\underline{x}_1^\sigma - \underline{y}) f^{\sigma}(\underline{x}_1^\sigma, \underline{v}_1^\sigma, t)}{\langle n_\sigma(\underline{y}, t) \rangle_F} + \\ &\quad \frac{1}{\langle n_\sigma(\underline{y}, t) \rangle_F} \int d\underline{x}_2^\sigma d\underline{v}_2^\sigma \delta(\underline{x}_2^\sigma - \underline{y}) f_2^{\sigma\sigma}(\underline{x}_1^\sigma, \underline{x}_2^\sigma, \underline{v}_1^\sigma, \underline{v}_2^\sigma, t) . \end{aligned} \quad 3.1.25$$

Similarly for all other species $\eta \neq \sigma$ we get:

$$\begin{aligned} f'^{\eta}(\underline{x}_1^\eta, \underline{v}_1^\eta, \tau; \underline{y}, t) \Big|_{\tau=0} &= N_\eta \int_{(\eta_1)} d'X \frac{\sum_{\sigma}^N \delta(\underline{x}_1^\sigma - \underline{y}) F(\underline{X}, t)}{\langle n_\sigma(\underline{y}, t) \rangle_F} \\ &= \frac{1}{\langle n_\sigma(\underline{y}, t) \rangle} N_\sigma N_\eta \int d\underline{x}_1^\sigma d\underline{v}_1^\sigma \delta(\underline{x}_1^\sigma - \underline{y}) \int_{(\sigma_1, \eta_1)} d''X F(\underline{X}, t) , \end{aligned} \quad 3.1.26$$

which in terms of the two particle distribution $f_2^{\eta\sigma}$ becomes:

$$f'^{\eta}(\underline{x}_1^\eta, \underline{v}_1^\eta, \tau; \underline{y}, t) \Big|_{\tau=0} = \int d\underline{x}_1^\sigma d\underline{v}_1^\sigma \delta(\underline{x}_1^\sigma - \underline{y}) f_2^{\eta\sigma}(\underline{x}_1^\eta, \underline{x}_1^\sigma, \underline{v}_1^\eta, \underline{v}_1^\sigma, t) . \quad 3.1.27$$

In the special case of a time stationary* and spatially homogeneous system the distribution functions f^η and $f_2^{\eta\sigma}$ are independent of time. The function f^η is

*

We include here systems, which are not necessarily in thermodynamic equilibrium but which are practically stationary in the time scales of interest, for instance, the case of electron and ions at different temperatures.

also independent of \underline{x}_1^η , and $f_2^{\eta\sigma}$ depends on \underline{x}_1^η and \underline{x}_1^σ through their difference $\underline{x}_1^\eta - \underline{x}_1^\sigma$. In this case we can write them in the form:

$$f^\eta(\underline{x}_1^\eta, \underline{v}_1^\eta, t) = n_\eta \Phi_\eta(\underline{v}_1^\eta) \quad \text{including } \eta = \sigma, \quad 3.1.28$$

$$f^{\eta\sigma}(\underline{x}_1^\eta, \underline{x}_1^\sigma, \underline{v}_1^\eta, \underline{v}_1^\sigma, t) = n_\eta n_\sigma \left[\Phi_\eta(\underline{v}_1^\eta) \Phi_\sigma(\underline{v}_1^\sigma) + p^{\eta\sigma}(\underline{x}_1^\eta - \underline{x}_1^\sigma, \underline{v}_1^\eta, \underline{v}_1^\sigma) \right], \quad 3.1.29$$

and

$$f^{\sigma\sigma}(\underline{x}_1^\sigma, \underline{x}_2^\sigma, \underline{v}_1^\sigma, \underline{v}_2^\sigma, t) = n_\sigma n_\sigma \left[\Phi_\sigma(\underline{v}_1^\sigma) \Phi_\sigma(\underline{v}_2^\sigma) + p^{\sigma\sigma}(\underline{x}_1^\sigma - \underline{x}_2^\sigma, \underline{v}_1^\sigma, \underline{v}_2^\sigma) \right]. \quad 3.1.30$$

The latter two equations are a convenient way of writing $f_2^{\eta\sigma}$ and $f^{\sigma\sigma}$. They can be taken as the definition of $p^{\eta\sigma}$ and $p^{\sigma\sigma}$. Here, n_η and n_σ are the constant densities of the η and σ species, and $\Phi_\eta(\underline{v})$ and $\Phi_\sigma(\underline{v})$ are velocity distribution functions normalized to unity.

In terms of above expressions for f^η , $f_2^{\eta\sigma}$ and $f_2^{\sigma\sigma}$, the initial conditions for the hypothetical system take a simpler form:

$$f'^\sigma(\underline{x}_1^\sigma, \underline{v}_1^\sigma, \tau) \Big|_{\tau=0} = \delta(\underline{x}_1^\sigma) \Phi_\sigma(\underline{v}_1^\sigma) + n_\sigma \Phi_\sigma(\underline{v}_1^\sigma) + n_\sigma \int d\underline{v}_2^\sigma p^{\sigma\sigma}(\underline{x}_1^\sigma, \underline{v}_1^\sigma, \underline{v}_2^\sigma) \quad 3.1.31$$

for the σ species and

$$f'^\eta(\underline{x}_1^\eta, \underline{v}_1^\eta, \tau) \Big|_{\tau=0} = n_\eta \Phi_\eta(\underline{v}_1^\eta) + n_\eta \int d\underline{v}_1^\sigma p^{\eta\sigma}(\underline{x}_1^\eta, \underline{v}_1^\eta, \underline{v}_1^\sigma) \quad \text{for all } \eta \neq \sigma. \quad 3.1.32$$

One gets rid of the \underline{y} dependance by selecting a system of coordinates in which $\underline{y} = 0$. The space-time density autocorrelation can be written in this case as $\rho_\sigma(\underline{r}, \tau)$ independent of \underline{y} and t , and equation 3.1.8 as:

$$\rho_\sigma(\underline{r}, \tau) = n_\sigma \langle n'_\sigma(\underline{r}, \tau) \rangle_{F'} \quad 3.1.33$$

We notice that in either case, stationary and homogeneous or not, the initial conditions for the hypothetical system are functionals of the two-particle distribution functions $f_2^{\sigma\sigma}$ and $f_2^{\eta\sigma}$ of the actual system. In the case of thermodynamic equilibrium they can be obtained in a relatively easy way by integrating from the equilibrium N-body distribution function (see Chapter IV, equations 4.1.5 and 4.1.6). In the general case, the problem of finding the two-particle distribution function (actually its integral over one of the velocity variables; a somewhat simpler problem) is perhaps the most difficult part of the problem. This problem has been treated elsewhere (see for instance, Duppre reference 26) and we will not elaborate on it here. We should mention, though, that under justifiable assumptions the two-particle distribution function can be expressed as a functional of the one-particle distribution functions f^η ; therefore, the space-time density autocorrelation function is fully specified given only the one-particle distribution functions at a given time ($t = 0$).

The thermodynamical equilibrium case is treated in more detail in the following chapter, which should also serve as an illustration of the use of the theorem.

CHAPTER IV

THE THERMODYNAMIC EQUILIBRIUM CASE — FORMULATION OF THE MATHEMATICAL PROBLEM

1. The Kinetic Equations and Their Initial Conditions.

In contrast with the generality of the previous chapters, from here on we shall restrict ourselves to the case of a homogeneous plasma in thermodynamical equilibrium and in a constant magnetic field.

Although we shall include the effects of Coulomb collisions, the model will restrict us to the case where the mean free path ℓ is larger than the reciprocal, λ , of the wave number k . In the case of the incoherent scatter \underline{k} is the same wave number as defined in 2.1.6; in the general fluctuation problem the wave number of the k^{th} component of the spatial Fourier transform of $\rho(\underline{r}, \tau)$. If it were not for the presence of the magnetic field, the condition $\ell/\lambda \gg 1$ would be in general the condition for the complete omission of the collision terms. As we discussed in the introduction and we shall see from our solution, this is not generally the case in the presence of a magnetic field.

We shall also make the usual assumption $\epsilon = 1/n_e h^2 \ll 1$ ($\epsilon \leq 5 \times 10^{-5}$ in the ionosphere) which most plasmas satisfy. Here, h is the Debye length defined as

$$h^{-2} = \sum_{\mu=1}^s \frac{4\pi n_{\mu} Z_{\mu}^2 e^2}{KT}, \text{ where } K \text{ is the Boltzman constant, } T \text{ the temperature,}$$

n_μ the density of species μ , e the charge of an electron, and Z_μ the charge per particle of species μ in electron charge units, negative for negative charges.*

With the help of the equivalence theorem and kinetic theory of plasmas we have reduced our problem of finding the electron density space-time autocorrelation ρ_e to that of finding the time evolution of an electron one-particle distribution function $f'^e(\underline{x}, \underline{v}, \tau)$ from initial conditions 3.1.31 and 3.1.32. Or, equivalently the solution of the set of s coupled integro-differential equations

$$\frac{\partial f'^\mu}{\partial \tau} + \underline{v} \cdot \frac{\partial f'^\mu}{\partial \underline{x}} + \left(\underline{v} \times \underline{B} \frac{Z_\mu e}{m_\mu c} + \frac{\underline{E}(\underline{x}, \tau)}{m_\mu} \right) \cdot \frac{\partial f'^\mu}{\partial \underline{v}} = \sum_{\eta=1}^s C(f'^\eta, f'^\mu) \quad 4.1.1$$

for all species μ , where

$$\underline{E}(\underline{x}, \tau) = \sum_{\eta=1}^s \int d\underline{v}' d\underline{x}' \frac{\partial}{\partial \underline{x}} \Psi_\eta(\underline{x} - \underline{x}') f'^\eta(\underline{x}', \underline{v}', \tau), \quad 4.1.2$$

and initial conditions as given by 3.1.31 and 3.1.32. Here, m_μ stands for the mass of species μ , c is the speed of light, \underline{B} the intensity of the magnetic field and $\partial/\partial \underline{x} \Psi_\eta(\underline{x} - \underline{x}')$ the field at \underline{x} due to a particle of species η at \underline{x}' ; in this case the Coulomb field:

$$\frac{\partial \Psi_\eta}{\partial \underline{x}} = Z_\eta e \frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3}. \quad 4.1.3$$

The terms $C(f'^\mu, f'^\eta)$ stand for the collisional terms and we shall come back to them shortly.

*

We are in general interested in $Z_\mu = 1$. It is convenient to keep Z_μ even in such a case as a way of carrying the sign. Electrons and ions are then treated alike.

In the case of thermodynamic equilibrium we can obtain expressions for f^μ , $f_2^{\mu\mu}$ and $f_2^{\eta\mu}$, needed in our expression for the initial conditions of f'^μ in 3.1.31 and 3.1.32, from equilibrium statistical mechanics. The function $\Phi(\underline{v})$ is then the maxwellian distribution

$$\Phi^\mu(\underline{v}) = \varphi^\mu(\underline{v}) \equiv \left(\frac{m_\mu}{2\pi KT} \right)^{3/2} \exp \left(-v^2 / (2KT/m_\mu) \right), \quad 4.1.4$$

and the two-particle distribution function $f_2^{\mu\eta}(\underline{x}, \underline{\bar{x}}, \underline{v}, \underline{\bar{v}})$, or, equivalently the function $P^{\mu\eta}(\underline{x} - \underline{\bar{x}}, \underline{v}, \underline{\bar{v}})$, can be obtained by integrating the N-body probability distribution function $F(\underline{X})$ which in thermodynamic equilibrium is given by:

$$F(\underline{X}) = \frac{e^{-H(\underline{X})/KT}}{\int d\underline{X} e^{-H(\underline{X})/KT}}. \quad 4.1.5$$

$H(\underline{X})$ being the Hamiltonian of the system. Guernsey, following such a procedure (see Appendix A of Montgomery and Tidman²⁷) obtains the well-known expression

$$P_{\mu\eta}(\underline{x} - \underline{\bar{x}}, \underline{v}, \underline{\bar{v}}) = \frac{Z_\mu Z_\eta e^2}{KT} \frac{e^{-|\underline{x} - \underline{\bar{x}}|/\hbar}}{|\underline{x} - \underline{\bar{x}}|} \varphi^\mu(\underline{v}) \varphi^\eta(\underline{\bar{v}}), \quad 4.1.6$$

which is valid for most values of the argument $|\underline{x} - \underline{\bar{x}}|$ with the exception of a very small volume with radius of the order of $r_e = e^2/KT$ (in terms of the small parameter ϵ and length \hbar , $r_e = \epsilon\hbar$). Equation 4.1.6 is also valid for $\mu = \eta$. Substituting 4.1.6 in 3.1.31 and 3.1.32 we obtain the following expressions for the initial conditions

$$f'^e(\underline{x}, \underline{v}, \tau) \Big|_{\tau=0} = \left(n_e + \delta(\underline{x}) - n_e \frac{e^2}{KT} \frac{e^{-|\underline{x}|/\lambda}}{|\underline{x}|} \right) \varphi^e(\underline{v}) \quad 4.1.7$$

for the electrons and

$$f'^\mu(\underline{x}, \underline{v}, \tau) \Big|_{\tau=0} = \left(n_\mu + n_\mu Z_\mu \frac{e^2}{KT} \frac{e^{-|\underline{x}|/\lambda}}{|\underline{x}|} \right) \varphi^\mu(\underline{v}) \quad 4.1.8$$

for all the ion species ($\mu \neq e$).

Figure 2 is a schematic representation of the initial density distribution as given by the parenthesis in 4.1.7 and 4.1.8. It gives a physical picture of the type of initial value problem we are trying to solve. We can interpret physically such a configuration as that produced by the presence of one electron (the delta function) which has been there for a sufficiently long time to produce an equilibrium configuration (just as in the classical Debye problem); with a hole of electrons and a cusp of ions in the immediate vicinity of the disturbing electron. At $t=0$ we let the disturbance go with velocity distributions $\varphi^e(\underline{v})$ and $\varphi^\mu(\underline{v})$. We are interested in the time evolution of such a configuration and, for the plane wave scattering problem, in the k^{th} component of its spatial Fourier transform.

2. The Linearized Equation.

We can linearize equation 4.1.1. Following the standard perturbation procedure we write

$$f'^\mu(\underline{x}, \underline{v}, \tau) = n^\mu \varphi^\mu(\underline{v}) + f_1^\mu(\underline{x}, \underline{v}, \tau) \quad \text{for all } \mu \text{'s}, \quad 4.2.1$$

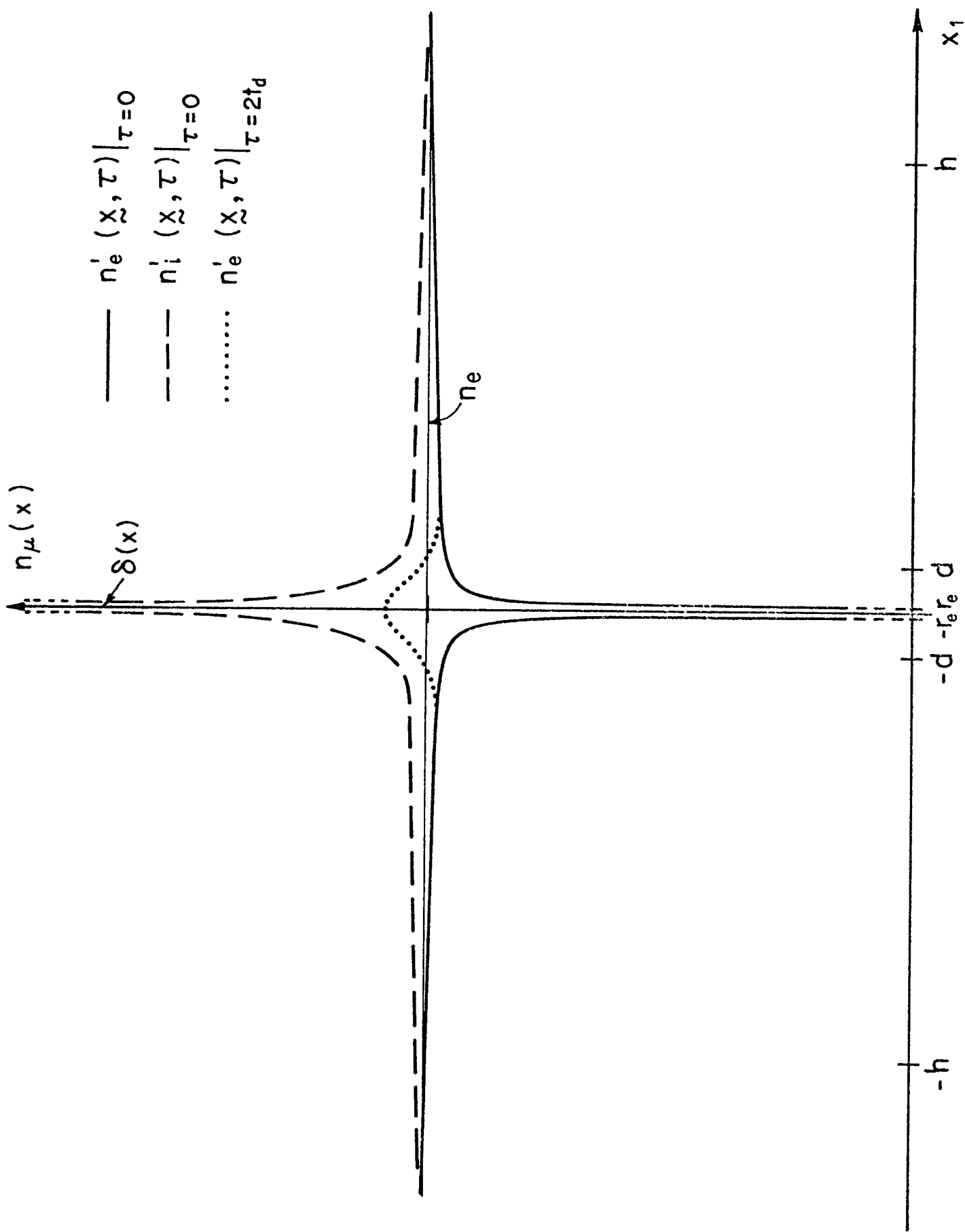


FIG. 2. Sketch of the initial value densities of the hypothetical system as a function of $\tilde{x} = (x_1, 0, 0)$

where $f_1^\mu(\underline{x}, \underline{v}, \tau)$ is assumed to be small with respect to $n^\mu \varphi^\mu(\underline{v})$.

Note we have dropped the prime convention for $f_1^\mu(\underline{x}, \underline{v}, \tau)$. This is to keep notation simple since $f_1^\mu(\underline{x}, \underline{v}, \tau)$ is the most used function from here on.

We define $n_1^\mu(\underline{x}, \tau)$ as the density of the perturbation, i.e.:

$$n_1^\mu(\underline{x}, \tau) = \int d\underline{v} f_1^\mu(\underline{x}, \underline{v}, \tau) . \quad 4.2.2$$

In terms of this density

$$\langle n^\mu(\underline{x}, \tau) \rangle = n^\mu + n_1^\mu(\underline{x}, \tau) . \quad 4.2.3$$

If the system is stable and $f_1^\mu(\underline{x}, \underline{v}, \tau)$ is small at an initial time it will be small at all later times. It would appear from the expression 4.1.7, with a delta function at $\underline{x}=0$, that this is not the case; but the total charge is only that of one electron, and in any case after a very small time $t_d = (n^e)^{-1/3}/u$, * the time it takes a thermal electron to travel an interparticle distance $d = (n^e)^{-1/3}$, the delta function would have dispersed to a Gaussian looking pip with an average amplitude of the order of n_e . At a time, let us say, equal to $3t_d$ it would have already a small amplitude of the order of $1/27 n_e$. The regions on the "cusp" and "hole" within a radius of order r_e are even less important since it can be shown they contain $\approx \epsilon^2$ particles and smear out equally fast.

In terms of the perturbations $f_1^\mu \equiv f_1^\mu(\underline{x}, \underline{v}, \tau)$ the set of equation 4.1.1 and 4.1.2 take the form:

* In terms of our small parameter, ϵ , and the reciprocal of the plasma frequency $\omega_e^{-1} = (4\pi n_e e^2/m_e)^{-1}$, we can write $t_d = \epsilon^{1/3} \omega_e^{-1}$.

$$\begin{aligned}
& \frac{\partial f_1^\mu}{\partial \tau} + \underline{v} \cdot \frac{\partial f_1^\mu}{\partial \underline{x}} + \underline{v} \times \underline{\Omega}_\mu \cdot \frac{\partial f_1^\mu}{\partial \underline{v}} + \sum_{\eta}^S \omega_\mu^2 \frac{Z_\eta}{\eta} \int d\underline{x}' d\underline{v}' \underline{\mathcal{G}}(\underline{x} - \underline{x}') f_1^\eta(\underline{x}', \underline{v}', \tau) \cdot \frac{\partial \varphi^\mu(\underline{v})}{\partial \underline{v}} \\
& = \sum_{\eta}^S \left[C(n_\eta \varphi^\eta, f_1^\mu) + C(f_1^\eta, n_\mu \varphi^\mu) \right]
\end{aligned} \tag{4.2.4}$$

and initial conditions

$$f_1^e(\underline{x}, \underline{v}, \tau) \Big|_{\tau=0} = \left(\delta(\underline{x}) - n_e \frac{e^2}{KT} \frac{e^{-|\underline{x}|/n}}{|\underline{x}|} \right) \varphi^e(\underline{v}) \tag{4.2.5}$$

for the electron and

$$f_1^\mu(\underline{x}, \underline{v}, \tau) \Big|_{\tau=0} = \left(n_\mu \frac{Z_\mu}{\eta} \frac{e^2}{KT} \frac{e^{-|\underline{x}|/n}}{|\underline{x}|} \right) \varphi^\mu(\underline{v}) \tag{4.2.6}$$

for the ions ($\mu \neq e$). The parameters ω_μ and $\underline{\Omega}_\mu$ are the plasma and gyro-frequencies of component μ , defined by $\omega_\mu^2 = 4\pi n_\mu Z_\mu^2 e^2 / m_\mu$ and $\underline{\Omega}_\mu = B Z_\mu e / m_\mu$. The function $\underline{\mathcal{G}}(\underline{x} - \underline{x}')$ is defined as $\underline{\mathcal{G}}(\underline{x}) = \underline{x} / 4\pi |\underline{x}|^3$.

We have assumed that $\sum_{\eta}^S C(n_\eta \varphi^\eta, n_\mu \varphi^\mu) = 0$ for all μ 's, a condition that any acceptable collision operator should satisfy since φ^η is the equilibrium distribution. We have also dropped all second order terms including $C(f_1^\mu, f_1^\eta)$.

We shall not take Fourier transform at this stage, but having linearized the equation and being interested particularly in the k^{th} component of the Fourier transform $\langle \bar{n}_e(\underline{\xi}, \tau) \rangle$ of $\langle n_e'(\underline{x}, t) \rangle$, it is convenient to keep in mind that this is only determined by the k^{th} component of the Fourier transform of the initial conditions of the different constituents. So, for the scattering application, one can think of

our problem as being that of finding the time evolution of an electron density wave (with wave vector \underline{k}) which initially has been set up, together with density waves of equal \underline{k} for the other constituents, to have relative amplitudes given by the Fourier transforms of their initial densities and to have velocity distributions $\varphi^e(\underline{v})$ and $\varphi^\mu(\underline{v})$.

3. The Collision Model.

We shall discuss now the collision model that we intend to use. We shall restrict ourselves for a while, for the sake of simplicity, to the case of one ion species and the electrons. As we mentioned in the introduction we shall use an approximate model of the Fokker-Planck type which can be considered as an appropriate approximation of the one proposed by Dougherty⁶, generalized to a multicomponent plasma and valid for the case $\lambda/\ell \ll 1$.

The inclusion of the electrons should not change the collision term for the ions appreciably and we could write it just as the single component model proposed by Dougherty. From 1.1.5, using the same notation used there ($f = f_0 + f_1$), and after differentiation of f_0 (a Maxwellian) we have

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = v \left[\frac{\partial}{\partial \underline{v}} \cdot \underline{v} f_1 + u^2 \frac{\partial^2}{\partial \underline{v} \partial \underline{v}} f \right] + v \left[\underline{\nabla} \cdot \frac{\underline{v}}{u^2} + \frac{T_1}{T_0} \left(\frac{v^2}{u^2} - 3 \right) \right] f_0, \quad 4.3.1$$

where $u^2 = KT_0/m$.

Our point now is that unless one is interested in scale sizes in the hydrodynamic range, i.e., for wavelengths larger than the mean free path ($\lambda \gg \ell$; $\ell = u v^{-1}$),

one can neglect the second parenthesis in above expression but not necessarily the first one.

We can roughly estimate the order of magnitude of the second parenthesis. Since the distribution function f_0 is almost zero for velocities which are just a few times larger than u we can write $(v/u) f_0 \leq f_0$ and $(v^2/u^2) f_0 \leq f_0$. We can say the same for f_1 if initially it was not populated for values of v a few times larger than u (as in our case), we expect at later times to conserve this same feature so from the definitions of \bar{V} and T_1 we can estimate that

$$\frac{|\bar{V}|}{u} = \left| \frac{1}{n_0} \int_u^v \tilde{f}_1 dv \right| \sim \frac{n_1}{n_0}$$

4.3.2

and

$$\frac{T_1}{T_0} = \frac{1}{3n_0} \int \frac{v^2}{u^2} \tilde{f}_1 dv \sim \frac{n_1}{n_0},$$

where

$$n_1 \equiv n_1(\underline{x}, t) = \int \tilde{f}_1(\underline{x}, \underline{v}, t) d\underline{v}.$$

4.3.3

So the second parenthesis in 4.3.1, behaves roughly as

$$v \left[\frac{\bar{V}}{u} \cdot \frac{v}{u^2} + \frac{T_1}{T_0} \left(\frac{v^2}{u^2} - 3 \right) \right] f_0 \sim v \frac{n_1(\underline{x}, t)}{n_0} f_0(v).$$

4.3.4

What we have said about $f_1(\underline{x}, \underline{v}, t)$ holds true for its Fourier transform $\bar{f}_1(\underline{\xi}, v, t)$, provided we replace its density $n_1(\underline{x}, t)$ by its Fourier transform $\bar{n}_1(\underline{\xi}, t)$. We are interested in values of $\underline{\xi} = \underline{k}$ (\underline{k} as defined in 2.1.5), and for what follows it is convenient to think of \bar{f}_1 and \bar{n}_1 as single wave perturbations.

Now, collision terms are rate terms and their total contribution to a change

in $\bar{f}_1(\underline{k}, \underline{v}, t)$ is given by their time integrated expression. If $\bar{n}_1(\underline{k}, t)$ were at all times of the same order of magnitude as its initial value $\bar{n}_1(\underline{k}, 0)$ then its integrated contribution after a time T would be of the order of $vT(\bar{n}_1(\underline{k}, 0)/n_0) f_0$ or of the order vT with respect to $\bar{f}_1(\underline{k}, \underline{v}, t)$ since $(\bar{n}_1(\underline{k}, 0)/n_0) f_0$ is of the order $\bar{f}_1(\underline{k}, \underline{v}, t)$. But as we shall see (which can be taken as an a posteriori justification), the ion gas in the neutralizing background of the electrons behaves almost like a gas of free-streaming non-interacting neutral particles, with very little collective Coulomb interaction, and in about the time it takes a thermal ion with velocity u to go from the crest to the trough of the initial wave perturbation, the amplitude of the density wave $\bar{n}_1(\underline{k}, t)$, as well as that of other moments, goes down to negligible values. It is this time, i.e., the life time of $\bar{n}_1(\underline{k}, t)$ which one should take in estimating the integrated magnitude of 4.3.4.

There are two cases to consider depending if the gyro-radius $r_g = u/\Omega$ is larger or smaller than the wavelength, $\lambda = 2\pi/k$.

In the first case the ion trajectories, for scale sizes of the order of half a wavelength, can be taken as straight lines and the life time of $\bar{n}_1(\underline{k}, t)$ would then be of the order of $\lambda/2u$ (the time it takes a thermal particle to travel half a wavelength) and therefore the second collision parenthesis would be of order $v\lambda/2u = \lambda/2\ell$ which we have assumed small. Later we shall see that due to the spiraling motion of the ions, and under certain conditions, it is possible for $\bar{n}_1(\underline{k}, t)$, at a time multiple of a gyro-period, to come back to appreciable amplitudes, but again only a short time of the order λ/u .

In the second case the ion trajectories are roughly that of the center of gyration constrained to move along the magnetic field. The life time of $\bar{n}_1(\underline{k}, t)$ would then be of order $\lambda/2u \cos \theta$, where θ is the angle between \underline{k} and \underline{B} , and the magnitude of the second collision parenthesis of order $\lambda/2\ell \cos \theta$. We see there that these terms can also be neglected provided the angle θ does not get too close to perpendicular, and at any angle provided one does not carry out the solution to times larger or of the order of $\lambda/2u \cos \theta$. Since we are not going to keep such collision terms, we exclude this region (time scales $\geq \lambda/2u \cos \theta$ at angles $\alpha = (\pi/2) - \theta \leq \sin^{-1} \lambda/2\ell$) from our solution. The solution would be valid for gyro-frequency detail at all angles since in this second case the gyro-period $t_g = 2\pi r_g/u$ would always be less than $\lambda/2u \cos \theta$ since we are assuming $r_g \ll \lambda$.

The first case, $\lambda \ll r_g$, is the most important for us, since for ionospheric incoherent scattering applications this is usually the case. At Jicamarca, for instance, $\lambda = 3$ meters, $r_g \approx 15$ meters for 0_+ and 0.3 gauss, and $\ell \geq 200$ meters.

Although the first collision term, the one of the Fokker-Planck type, is also multiplied by the same relatively small ν as the term we have just discussed, it can not be neglected under similar conditions, especially in the $r_g \gg \lambda$ case. Since we are going to keep such a term, the best arguments will be given by analyzing our solution. In any case, we can say at this time that this is due to the fact that this term depends on the derivatives of f_1 and not on its short lived moments. In fact, under certain conditions, its integrated contribution is larger than νT , where T is the integration time. This is due to a behavior of νf_1 of the form $\nu e^{-i\mathbf{k} \cdot \underline{v} t} \varphi(\underline{v})$

which becomes very spiky as time progresses, and upon double differentiation with respect to \underline{v} gives terms of the form $\nu k^2 t^2 e^{-ik \cdot \underline{v} t} \varphi(\underline{v})$ which in turn integrates in time to terms of order $(\nu T)(k^2 u^2 T^2)$. For any given value of νT there will always be a \underline{k} for which this term becomes important.

We can generalize our discussion to the case of several ions. After linearization we can still divide the collision term for each component into two terms, one as before, consisting of the Fokker-Planck operator operating on the perturbation f_1 of the distribution function in question, and the other proportional to its unperturbed distribution f_0 multiplied by first and second moments of the perturbations of the distribution functions of all other components. The arguments about the order of magnitude of both terms hold, so we neglect the second term and keep only the first one with the Fokker-Planck operator. The only difference now between the single ion and the multiple ion case is the value of ν which is now a functional of all the unperturbed distribution functions.

For the electron equation we can in general neglect all collision terms, the reason now being the dominance of the self-consistent field. We shall still keep the Fokker-Planck term, more for the reason of convenience than anything else, since there is certain advantage in keeping the equations for all of the components alike; besides which we can use it as a warning flag to indicate to us any possible condition in which this term may become important. In order to estimate the magnitude of the collision terms being neglected in the electron equation, it is convenient to show explicitly two time scales in the dynamics of the electrons, a fast one corre-

sponding to the electron dynamics with the ions stationary, and a slow one corresponding to the dynamics of the ions, which the electrons follow adiabatically. This can be done in a formal way by making a multitime-scale expansion using two times t and $T = \delta t$, where $\delta = (m_e/m_i)^{1/2}$.

Let $\bar{f}_1^e(\underline{k}, \underline{v}, t)$ and $\bar{f}_1^i(\underline{k}, \underline{v}, t)$ be the perturbations (from linearization) with wave number \underline{k} for the distribution functions of electrons and ions respectively.

We can write

$$\bar{f}_1^e(\underline{k}, \underline{v}, t) = \bar{f}_{1,0}^e(\underline{k}, \underline{v}, t, T) + \delta \bar{f}_{1,1}^e(\underline{k}, \underline{v}, t, T) + O(\delta^2) \quad 4.3.5$$

for the electrons and

$$\bar{f}_1^i(\underline{k}, \underline{v}, t) = \bar{f}_{1,0}^i(\underline{k}, \underline{v}, t, T) + \delta \bar{f}_{1,1}^i(\underline{k}, \underline{v}, t, T) + O(\delta^2) \quad 4.3.6$$

for the ions (we consider one ion species for simplicity).

For the sake of estimating the magnitude of the collision terms we can take the zeroth order terms as being sufficiently accurate. To zeroth order in δ one can show (by inserting 4.3.5 and 4.3.6 into their corresponding equations and neglecting higher order terms) that: $\partial/\partial t \bar{f}_{1,0}^i(\underline{k}, \underline{v}, t, T) = 0$; that is that the ions evolve in the time scale T only, and that the electrons evolve according to

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \underline{k} \cdot \underline{v} + \underline{v} \times \underline{\Omega} \cdot \frac{\partial}{\partial \underline{v}} \right) \bar{f}_{1,0}^e(\underline{k}, \underline{v}, t, T) + \left[\bar{n}_{1,0}^e(\underline{k}, t, T) - n_{1,0}^i(\underline{k}, T) \right] \frac{\omega_e^2}{k^2 u_e^2} i \underline{k} \cdot \underline{v} \varphi^e(\underline{v}) \\ & = v_e \left(\frac{\partial}{\partial \underline{v}} \underline{v} + u^2 \frac{\partial}{\partial \underline{v} \partial \underline{v}} \right) \bar{f}_{1,0}^e(\underline{k}, \underline{v}, t, T) + \left(v_{ee} \mathcal{M}^e(t, T) + v_{ei} \mathcal{M}^i(T) \right) \varphi^e(\underline{v}). \end{aligned} \quad 4.3.7$$

Here, $u_e = kT/m_e$, $\mathcal{M}^e(t, T)$ and $\mathcal{M}^i(T)$ stand for some moments of the electron and ion distribution functions and are of order $\bar{n}_{1,0}^e(\underline{k}, t, T)$ and $\bar{n}_{1,0}^i(\underline{k}, T)$ respectively, similar to the terms we discussed in more detail for the simple ion case; ν_{ee} and ν_{ei} have dimensions sec^{-1} and correspond to the reciprocal electron-electron and electron-ion relaxation times. The coefficient $(\omega_e^2/k^2 u_e^2) \underline{k} \cdot \underline{v}$ has also dimensions of sec^{-1} and it is of order $(\omega_e/k u_e) \omega_e$, where ω_e is the electron plasma frequency; in terms of the Debye length it can be written as $(\omega_e/k u_e) \omega_e = (1/2k^2 h^2) k u_e$. Note that as far as t goes, one can consider all ion terms as independent inhomogeneous constant terms. This is the reason for carrying the two times scale expansion.

There are two cases to consider: either $1/k^2 h^2$ is smaller or larger than unity (that is, the wavelength (actually $\lambda/2\pi$) is smaller or larger than the Debye length h). In the first case the self-consistent field is not an important term and the electron gas has a diffusive behaviour (the $\partial/\partial t + \underline{k} \cdot \underline{v} + \underline{v} \times \underline{\Omega} \cdot \partial/\partial \underline{v}$ terms predominate) just as the ion gas which we have just considered, with the important difference that in this case $\nu/k u_e$ or equivalently λ/ℓ would be very small* and so would the integration times. Our arguments given before for the smallness of the collision terms other than the Fokker-Planck term also apply in this case. The second case in which $1/k^2 h^2 \gg 1$ is more important for ionospheric application since in the ionospheric Debye length is of the order of a few millimeters to a few centimeters

*

Since we are assuming $\lambda < h$, then $\lambda/\ell < h/\ell$. The order of h/ℓ can be estimated as a function of the small parameter ε . Taking Spitzer's formula $\nu_{ee}^{-1} = m_e^{1/2} (3KT)^{3/2} / 8 \times 0.714 \times \pi n_e e^2 \ln \Lambda$, where $\Lambda = 3/2 e^3 K^3 T^3 / \pi n_e$ one can show that $h/\ell = o(\varepsilon \ln \varepsilon)$.

and the wavelengths used (so far) are larger than that. In this case we group all ion terms and all electron moment terms on 4.3.7 together, with $\varphi^e(\underline{v})$ as a common factor, and write:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \underline{k} \cdot \underline{v} + \underline{v} \times \underline{\Omega} \cdot \frac{\partial}{\partial \underline{v}} - \underline{v} \frac{\partial}{\partial \underline{v}} \cdot \underline{v} - \nu_e \frac{\partial^2}{\partial \underline{v} \cdot \partial \underline{v}} \right) f_{1,o}^e(\underline{k}, \underline{v}, t, T) \\ & + \left(\frac{\underline{k} \cdot \underline{v}}{2k^2 h^2} \bar{n}_{1,o}^e(\underline{k}, t, T) - \nu_{ee} \mathcal{M}^e(t, T) \right) \varphi^e(\underline{v}) + \left[\frac{\underline{k} \cdot \underline{v}}{2k^2 h^2} \bar{n}_{1,o}^i(\underline{k}, T) + \nu_{ei} \mathcal{M}^i(T) \right] \varphi^e(\underline{v}). \end{aligned}$$

4.3.8

One can clearly see then that the ion collision terms compare to the ion Coulomb field terms as $(\nu_{ei}/ku)(k^2 h^2)$ and can be neglected, since both ν_{ei}/ku and $k^2 h^2$ are small factors. Similarly we can neglect the electron collision terms, involving moments of $\bar{f}_{o,1}^e$ compared with the electron Coulomb field term. Notice that, for the electron equation, if $1/k^2 h^2$ is sufficiently large we can neglect collisions even in the case that $\nu_{ei}/ku > 1$, that is, even in cases where the mean free path is less than the wavelength.

We shall emphasize that although the ion equation also has the large $1/k^2 h^2$ factor we cannot use it as an argument for estimating order of magnitudes of the different terms. The difference is that the ions evolve mainly in the T-scale, and in this scale the two densities $\bar{n}_{1,o}^e(\underline{k}, t, T)$ and $\bar{n}_{1,o}^i(\underline{k}, T)$ are not decoupled from each other as they are in the electron equation for $f_{1,o}^e(\underline{k}, \underline{v}, t, T)$; in fact they almost cancel each other. This cancelling we have referred to before as the neutralizing effect of the electrons.

In terms of the approximate collision operator proposed and discussed above the problem has been reduced to finding the solution of the following set of s coupled integro-differential equations:

$$\begin{aligned} \frac{\partial f_1^\mu}{\partial t} + \underline{v} \cdot \frac{\partial f_1^\mu}{\partial \underline{x}} + \underline{v} \times \underline{\Omega} \cdot \frac{\partial f_1^\mu}{\partial \underline{v}} - \underline{v}_\mu \frac{\partial}{\partial \underline{v}} \underline{v} f_1^\mu - \underline{v}_\mu u_\mu^2 \frac{\partial^2}{\partial \underline{v} \partial \underline{v}} f_1^\mu \\ = \sum_{\eta=1}^s \omega_\mu^2 \frac{Z_\eta}{Z_\mu} \int d\underline{x}' d\underline{v}' \delta(\underline{x} - \underline{x}') f_1^\eta(\underline{x}', \underline{v}', t) \cdot \frac{\partial \varphi^\mu(\underline{v})}{\partial \underline{v}}, \quad \text{for all } \mu\text{'s}, \end{aligned}$$

4.3.9

with initial conditions as given by 4.2.5 and 4.2.6. We are especially interested in the k^{th} component of the Fourier transform $\bar{n}_1^e(\underline{\xi}, t) \Big|_{\underline{\xi}=\underline{k}}$ of the perturbation density of the electrons $n_1^e(\underline{x}, t)$.

To specify our problem completely we need the values of \underline{v}_μ as functionals of the system distribution functions $n_\mu \varphi^\mu(\underline{v})$. These are obtained in Appendix I by taking our Fokker-Planck operator with constant \underline{v}_μ as an approximation of the more accurate Rosenbluth, MacDonald and Judd³² Fokker-Planck equation where \underline{v}_μ is a \underline{v} dependent functional of the system distribution functions.

CHAPTER V

SOLUTION OF THE INITIAL VALUE PROBLEM

1. Solution in Terms of Green's Functions.

We are interested in solving the set of equations 4.3.9 which we write in the form

$$\mathcal{L}^\mu f_1^\mu(\underline{x}, \underline{v}, t) = \omega_\mu^2 \sum_{\eta=1}^S \frac{Z_\eta}{Z_\mu} \int d\underline{x}' d\underline{v}' f_1^\eta(\underline{x}', \underline{v}', t) \delta(\underline{x} - \underline{x}') \cdot \underline{\varphi}'(\underline{v}'), \quad 5.1.1$$

where \mathcal{L}^μ stands for the operator operating on f_1^μ on the left hand side of 4.3.9 and $\underline{\varphi}'(\underline{v})$ stands for $\partial \underline{\varphi}^\mu(\underline{v}) / \partial \underline{v}$; with initial conditions 4.2.5 and 4.2.6 which we write as

$$f_1^\mu(\underline{x}, \underline{v}, t) \Big|_{t=0} = J^\mu(\underline{x}, \underline{v}) = H^\mu(\underline{x}) \varphi^\mu(\underline{v}), \quad 5.1.2$$

where $H^\mu(\underline{x})$ stand for the velocity independent factors:

$$H^e(\underline{x}) = \delta(\underline{x}) - n_e \frac{e^2}{KT} \frac{e^{-|\underline{x}|/\hbar}}{|\underline{x}|} \quad \text{for } \mu = e \quad 5.1.3$$

and

$$H^\mu(\underline{x}) = n_\mu \frac{Z_\mu e^2}{KT} \frac{e^{-|\underline{x}|/\hbar}}{|\underline{x}|} \quad \text{for } \mu \neq e. \quad 5.1.4$$

Using standard Green's function techniques, we can convert the set of coupled integro-differential equations 5.1.1 into a set of plain (coupled) integral equations.

Let $g^\mu(\underline{x}-\underline{x}_0, \underline{v}, \underline{v}_0, t-t_0)$ be the Green's function of \mathcal{L}^μ , that is, the solution of

$$\mathcal{L}^\mu g^\mu(\underline{x}-\underline{x}_0, \underline{v}, \underline{v}_0, t-t_0) = \delta(\underline{x}-\underline{x}_0) \delta(\underline{v}-\underline{v}_0) \delta(t-t_0) \quad \text{for } t \geq t_0 \quad 5.1.5$$

and the "causality condition": $g^\mu(\underline{x}-\underline{x}_0, \underline{v}, \underline{v}_0, t-t_0) = 0$ for $t < t_0$.

Then we can write $f_1^\mu(\underline{x}, \underline{v}, t)$ as the solution of

$$\begin{aligned} f_1^\mu(\underline{x}, \underline{v}, t) = & \int d\underline{x}_0 d\underline{v}_0 g^\mu(\underline{x}-\underline{x}_0, \underline{v}, \underline{v}_0, t) J^\mu(\underline{x}_0, \underline{v}_0) + \int_0^t dt_0 \int d\underline{x}_0 d\underline{v}_0 \\ & \times g^\mu(\underline{x}-\underline{x}_0, \underline{v}, \underline{v}_0, t-t_0) \omega_\mu^2 \frac{Z_\eta}{Z_\mu} \sum_{\eta=1}^8 \int d\underline{x}' n_1^\eta(\underline{x}', t_0) \delta(\underline{x}_0 - \underline{x}') \varphi'^\mu(\underline{v}_0). \end{aligned} \quad 5.1.6$$

We have made explicit use of the invariance of the \mathcal{L}^μ operator under space and time displacement in writing the dependence of g^μ upon $\underline{x}, \underline{x}_0$ and t, t_0 through their difference $\underline{x}-\underline{x}_0$ and $t-t_0$.

Let $\hat{f}_1^\mu(\underline{\xi}, \underline{v}, z)$, $\hat{n}_1^\mu(\underline{\xi}, z)$, and $\hat{g}^\mu(\underline{\xi}, \underline{v}, \underline{v}_0, z)$ be the Fourier-Laplace transform of $f_1^\mu(\underline{x}, \underline{v}, t)$, $n_1^\mu(\underline{x}, t)$, and $g^\mu(\underline{x}, \underline{v}, \underline{v}_0, t)$ respectively defined by:

$$\hat{f}_1^\mu(\underline{\xi}, \underline{v}, z) = \int d\underline{x} \int_0^\infty dt f_1^\mu(\underline{x}, \underline{v}, t) e^{-i\underline{\xi} \cdot \underline{x} - izt} \quad 5.1.7$$

$$\hat{n}_1^\mu(\underline{\xi}, z) = \int d\underline{x} \int_0^\infty dt n_1^\mu(\underline{x}, t) e^{-i\underline{\xi} \cdot \underline{x} - izt} \quad 5.1.8$$

$$\hat{g}^\mu(\underline{\xi}, \underline{v}, \underline{v}_0, z) = \int d\underline{x} \int_0^\infty dt g^\mu(\underline{x}, \underline{v}, \underline{v}_0, t) e^{-i\underline{\xi} \cdot \underline{x} - izt} \quad 5.1.9$$

and $\bar{\mathcal{G}}(\underline{\xi})$ and $\bar{J}^\mu(\underline{\xi}, \underline{v})$ the Fourier transforms of $\mathcal{G}(\underline{x})$ and $J^\mu(\underline{x}, \underline{v})$ which we can readily evaluate from the defining expressions for $\mathcal{G}(\underline{x})$ and $J^\mu(\underline{x}, \underline{v})$, viz.,

$$\bar{\phi}(\xi) = \int d\tilde{x} \frac{\tilde{x}}{4\pi|\tilde{x}|^3} e^{-i\xi \cdot \tilde{x}} = i \frac{\xi}{|\xi|^2}, \quad 5.1.10$$

and

$$\bar{J}^\mu(\xi, \nu) = \varphi^\mu(\nu) \int d\tilde{x} H^\mu(\tilde{x}) e^{-i\xi \cdot \tilde{x}} = \varphi^\mu(\nu) \bar{H}^\mu(\xi); \quad 5.1.11$$

where

$$\bar{H}^e(\xi) = \int d\tilde{x} \left(\delta(\tilde{x}) - n_e \frac{e^2}{KT} \frac{e^{-|\tilde{x}|/\hbar}}{|\tilde{x}|} \right) e^{-i\xi \cdot \tilde{x}} = 1 - \frac{h_e^2}{h_e^2} \frac{1}{1+h_e^2 \xi^2} \quad 5.1.12$$

for the electrons and

$$\bar{H}^\mu(\xi) = \int d\tilde{x} \left(n_\mu \frac{Z_\mu e^2}{KT} \frac{e^{-|\tilde{x}|/\hbar}}{|\tilde{x}|} \right) e^{-i\xi \cdot \tilde{x}} = \frac{h_\mu^2}{h_\mu^2} \frac{1}{1+h_\mu^2 \xi^2} \quad 5.1.13$$

for all other species $\mu \neq e$; and where h_μ^2 is a sort of single component "Debye

length" defined as $h_\mu^2 = KT/4\pi N_\mu Z_\mu^2 e^2$ (note that $h^{-2} = \sum_{\mu=1}^S h_\mu^{-2}$ and also

$$h_\mu^2 = u_\mu^2 / \omega_\mu^2).$$

Taking the Fourier-Laplace transform of 5.1.6 and making use of the convolution theorem (The first integral with respect to \tilde{x}_0 is a convolution with respect to \tilde{x} , and the double integral with respect to \tilde{x}_0 and \tilde{x}' is a double convolution with respect to \tilde{x} . The time integral is also of the convolution type), we get

$$\begin{aligned} \hat{\Gamma}_1^\mu(\xi, \nu, z) = & \int d\nu_0 \hat{g}^\mu(\xi, \nu, \nu_0, z) \bar{J}(\xi, \nu_0) \\ & + \omega_\mu^2 \sum_{\eta=1}^S \frac{Z_\eta}{Z_\mu} \hat{n}_1^\eta(\xi, z) \bar{\phi}(\xi) \int d\nu_0 \hat{g}^\mu(\xi, \nu, \nu_0, z) \varphi'^\mu(\nu_0). \end{aligned} \quad 5.1.14$$

If we integrate the above equation over ν we get the following set of linear algebraic

equations in $\hat{n}_1^\mu(\xi, z)$:

$$\begin{aligned} \hat{n}_1^\mu(\xi, z) = & \int d\tilde{v} d\tilde{v}_o \hat{g}^\mu(\xi, \tilde{v}, \tilde{v}_o, z) \bar{J}^\mu(\xi, \tilde{v}_o) \\ & + \frac{\omega_\mu^2}{Z_\mu} \bar{\phi}(\xi) \int d\tilde{v} d\tilde{v}_o \hat{g}^\mu(\xi, \tilde{v}, \tilde{v}_o, z) \varphi'^\mu(\tilde{v}_o) \sum_{\eta=1}^s Z_\eta \hat{n}_1^\eta(\xi, z) . \end{aligned} \quad 5.1.15$$

Let us define two functions $\hat{I}^\mu(\xi, z)$ and $\hat{S}^\mu(\xi, z)$ as:

$$\hat{I}^\mu(\xi, z) = \int d\tilde{v} d\tilde{v}_o \hat{g}^\mu(\xi, \tilde{v}, \tilde{v}_o, z) \bar{J}^\mu(\xi, \tilde{v}_o) \quad 5.1.16$$

$$\hat{S}^\mu(\xi, z) = - \xi^2 h_\mu^2 \omega_\mu^2 \bar{\phi}(\xi) \int d\tilde{v} d\tilde{v}_o \hat{g}^\mu(\xi, \tilde{v}, \tilde{v}_o, z) \varphi'^\mu(\tilde{v}_o) . \quad 5.1.17$$

In terms of these functions, equations 5.1.15 become

$$\hat{n}_1^\mu(\xi, z) = \hat{I}^\mu(\xi, z) - \frac{1}{\xi^2 h_\mu^2 Z_\mu} \hat{S}^\mu(\xi, z) \sum_{\eta=1}^s Z_\eta \hat{n}_1^\eta(\xi, z) . \quad 5.1.18$$

Multiplying 5.1.18 by Z_μ and summing all s equations, we get an equation for

$\sum_{\eta=1}^s Z_\eta \hat{n}_1^\eta(\xi, z)$ alone, with solution:

$$\sum_{\eta=1}^s Z_\eta \hat{n}_1^\eta(\xi, z) = \frac{\sum_{\eta=1}^s Z_\eta \hat{I}^\eta(\xi, z)}{1 + \sum_{\eta=1}^s \frac{1}{\xi^2 h_\eta^2} \hat{S}^\eta(\xi, z)} . \quad 5.1.19$$

Inserting 5.1.19 into 5.1.15 and in 5.1.18 we get explicit expressions for $\hat{f}_1^\mu(\xi, \tilde{v}, z)$

and $\hat{n}_1^\mu(\xi, z)$:

$$\begin{aligned} \hat{f}_1^\mu(\underline{\xi}, \underline{v}, z) = & \int d\underline{v}_o \hat{g}^\mu(\underline{\xi}, \underline{v}, \underline{v}_o, z) \bar{J}^\mu(\underline{\xi}, \underline{v}_o) \\ & + \frac{\omega_\mu^2}{Z_\mu} \bar{g}(\underline{\xi}) \int d\underline{v}_o \hat{g}^\mu(\underline{\xi}, \underline{v}, \underline{v}_o, z) \varpi^\mu(\underline{v}_o) \frac{\sum_\mu Z_\mu \hat{I}^\eta(\underline{\xi}, z)}{1 + \sum_{\eta=1}^S \frac{1}{\xi^2 h^2} \hat{S}^\eta(\underline{\xi}, z)} \end{aligned} \quad 5.1.20$$

$$\hat{n}_1^\mu(\underline{\xi}, z) = \hat{I}^\mu(\underline{\xi}, z) - \frac{1}{Z_\mu \xi^2 h^2} \frac{\hat{S}^\mu(\underline{\xi}, z) \sum_{\eta=1}^S Z_\eta \hat{I}^\eta(\underline{\xi}, z)}{1 + \sum_{\eta=1}^S \frac{1}{\xi^2 h^2} \hat{S}^\eta(\underline{\xi}, z)} \quad 5.1.21$$

From equations 2.1.10 and 3.1.33 we can get an expression for the signal autocorrelation function $R(\tau) \equiv \langle E(t) E(t+\tau) \rangle$ in terms of the Fourier transform $\bar{n}_1^e(\underline{k}, z)$ of the perturbation $n_1^e(\underline{x}, \tau)$ of the hypothetical system, viz.,

$$R(\tau) \equiv \langle E(t) E(t+\tau) \rangle = \frac{1}{2} |A^2| n^e V \Re e^{-i\omega_o \tau} \bar{n}_1^e(\underline{\xi}, \tau) \Big|_{\underline{\xi}=\underline{k}} \quad 5.1.22$$

for $\tau > 0$

Therefore, as far as the incoherent scattering problem is concerned, we are only interested in $\bar{n}_1^e(\underline{\xi}, \tau)$. This is given by the Laplace inverse

$$\bar{n}_1^e(\underline{\xi}, \tau) = \int_{-\infty + i\sigma}^{+\infty + i\sigma} \hat{n}_1^e(\underline{\xi}, z) e^{iz\tau} dz \quad 5.1.23$$

of $\hat{n}_1^e(\underline{\xi}, z)$ as given by equation 5.1.21.

In writing 5.1.22 we have excluded the case $|\underline{k}| = 0$ in assuming that the Fourier transform of the unperturbed density n_e is zero. The case $|\underline{k}| = 0$ cor-

responds to the scattering in the direction of propagation and has been excluded from our treatment.

In order to complete our analytical solution for $\hat{n}_1^\mu(\underline{\xi}, z)$ we need to evaluate the integrals $\hat{I}^\mu(\underline{\xi}, z)$ and $\hat{S}^\mu(\underline{\xi}, z)$ in terms of the defining parameters of the medium; this in turn involves evaluation of the Green's functions, which is our following task.

2. Evaluation of the Green's Function.*

We are interested in the solution $g^\mu(\underline{x}, \underline{v}, \underline{v}_0, t)$ of equation 5.1.5 for the case $\underline{x}_0 = 0$, $t_0 = 0$. We shall drop for simplicity the superscript to be recovered later when needed. It can be shown that $g \equiv g(\underline{x}, \underline{v}, \underline{v}_0, t)$ is also a solution of the more convenient initial value problem

$$\mathcal{L}g(\underline{x}, \underline{v}, \underline{v}_0, t) = 0 \quad \text{for } t > 0 \quad 5.2.1$$

$$g(\underline{x}, \underline{v}, \underline{v}_0, t) \Big|_{t=0} = \delta(\underline{x}) \delta(\underline{v} - \underline{v}_0) . \quad 5.2.2$$

The operator \mathcal{L} is the expression on the left-hand side of 4.3.9. We can write equation 5.2.1 explicitly in the form

$$\frac{\partial g}{\partial t} + A_{ij} \frac{\partial^2 g}{\partial w_i \partial w_j} + B_{ij} w_i \frac{\partial g}{\partial w_j} + Cg = 0 , \quad 5.2.3$$

* This problem has been solved by Dougherty.⁶ We differ in that we take a Fourier transform in phase space with the corresponding advantages for our application discussed in the text. Both methods lead to equations 5.2.12, 5.2.13, and 5.2.14.

where w_i are the components of a (6-dimensional column vector $\underline{w} = \begin{bmatrix} \underline{x} \\ \underline{v} \end{bmatrix}$ with components $x_1, x_2, x_3, v_1, v_2, v_3$. A_{ij} are the components of a 6×6 dimensional matrix

$$\underline{A} = \begin{bmatrix} \underline{A}^{xx} & \underline{A}^{xv} \\ \underline{A}^{vx} & \underline{A}^{vv} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -v^2 u^2 I \end{bmatrix}, \quad 5.2.4a$$

where \underline{A}^{xx} , \underline{A}^{xv} , \underline{A}^{vx} , are null 3×3 submatrices and $\underline{A}^{vv} = -v^2 u^2 I$, I being the 3×3 identity matrix; and B_{ij} the components of a 6×6 matrix:

$$\underline{B} = \begin{bmatrix} \underline{B}^{xx} & \underline{B}^{xv} \\ \underline{B}^{vx} & \underline{B}^{vv} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I & \underline{B}^{vv} \end{bmatrix}, \quad 5.2.4b$$

where \underline{B}^{xx} and \underline{B}^{xv} are 3×3 null submatrices, $\underline{B}^{vx} = I$, and

$$\underline{B}^{vv} = \begin{bmatrix} -v & -\Omega_3 & \Omega_2 \\ \Omega_3 & -v & -\Omega_1 \\ -\Omega_2 & \Omega_1 & -v \end{bmatrix}. \quad 5.2.5$$

Here, $\Omega_1, \Omega_2, \Omega_3$ are the components of the pseudo vector $\underline{\Omega}$. Summation over repeated indexes is implied in 5.2.3 and what follows. Let $G(\underline{\alpha}, t; \underline{v}_0)$ be the Fourier transform in \underline{w} space of $g(\underline{w}, t; \underline{v}_0) \equiv g(\underline{x}, \underline{v}, \underline{v}_0, t)$ defined by

$$G(\underline{\alpha}, t; \underline{v}_0) = \int d\underline{w} e^{-i \alpha_i w_i} g(\underline{w}, t; \underline{v}_0), \quad 5.2.6$$

where α_i stands for the 6 components $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ of a 6-dimensional row vector $\underline{\alpha} = [\underline{\xi}, \underline{\eta}]$.

Operating with $\int d\tilde{w} e^{-i\alpha_i w_i}$ on equation 5.2.3 and initial condition

5.2.2 we get an equation for $G \equiv G(\alpha, t; \tilde{v}_0)$

$$\frac{\partial G}{\partial t} - \alpha_i A_{ij} \alpha_j G - B_{ij} \frac{\partial}{\partial \alpha_i} \alpha_j G + CG = 0 \quad 5.2.7$$

and the initial condition

$$G(\alpha, t; \tilde{v}_0) \Big|_{t=0} = e^{-i\alpha_i w_{oi}}, \quad 5.2.8$$

where w_{oi} are the components of a 6-dimensional column vector $\tilde{w}_0 = \begin{bmatrix} \tilde{x}_0 \\ \tilde{v}_0 \end{bmatrix}$ with $\tilde{x}_0 = \tilde{0}$.

The three dimensional vector matrices $\tilde{x}, \tilde{v}, \tilde{\xi}, \tilde{v}_0$ correspond to a particular representation of the same vectors we have introduced previously, and we will use the same symbol for the vector and matrix. We shall use for convenience a system of coordinates in which $\Omega_1 = \Omega$ and $\Omega_2 = \Omega_3 = 0$, i.e., one in which the "1" axis is aligned with magnetic field.

Before proceeding to solve equation 5.2.7, we would like to remark that 5.2.7 is not only simpler to solve than 5.2.3 (we have reduced the order of the equation by one) but since in our particular application we are interested mainly in $\hat{I}(\tilde{\xi}, z)$ and $\hat{S}(\tilde{\xi}, z)$, these functions can be evaluated more readily from $G(\alpha, t; \tilde{v}_0)$ than from the inverse $g(\tilde{x}, \tilde{v}, \tilde{v}_0, t)$. The functions $\hat{I}(\tilde{\xi}, z)$ and $\hat{S}(\tilde{\xi}, z)$ involve integrals of the form:

$$\int d\tilde{v} d\tilde{x} e^{-i\tilde{\xi}_i x_i} g(\tilde{x}, \tilde{v}, \tilde{v}_0, t),$$

which can be evaluated directly from $G(\underline{\alpha}, t; \underline{v}_0)$ by letting $\eta = 0$, since

$$G(\underline{\alpha}, t; \underline{v}_0) \Big|_{\underline{\eta}=0} = \int d\underline{v} d\underline{x} e^{-i \underline{\xi}_i \underline{x}_i} g(\underline{x}, \underline{v}, \underline{v}_0, t) . \quad 5.2.9$$

Let us assume a solution for 5.2.7 of the form

$$G(\underline{\alpha}, t; \underline{v}_0) = \exp \left(-K(t) - i q_k(t) \alpha_k - \frac{1}{2} \alpha_k \Lambda_{km}(t) \alpha_m \right) . \quad 5.2.10$$

Inserting it into 5.2.7 we find that it is indeed a solution if the following equation is satisfied for arbitrary $\underline{\alpha}$:

$$\begin{aligned} -\frac{\partial K}{\partial t} - i \alpha_i \frac{\partial q_i}{\partial t} - \frac{1}{2} \alpha_k \frac{\partial \Lambda_{km}}{\partial t} \alpha_m - \alpha_i A_{ij} \alpha_j + B_{ij} \alpha_j q_i \\ + \frac{1}{2} B_{ij} \alpha_j \Lambda_{im} \alpha_m + \frac{1}{2} B_{ij} \alpha_j \alpha_k \Lambda_{ki} - B_{ij} \delta_{ij} + C = 0 . \end{aligned} \quad 5.2.11$$

We can take Λ_{km} to be symmetric since an arbitrary anti-symmetric part in Λ_{km} does not contribute to the quadratic expression in 5.2.10. Then, to satisfy 5.2.11 for all $\underline{\alpha}$ the coefficients of each independent term of above quadratic expression in $\underline{\alpha}$ must be equal to zero. In matrix notation* we must have

$$-\frac{dK}{dt} = \text{Tr} \underset{\approx}{B} - C = 0 \quad 5.2.12$$

(since $\text{Tr} \underset{\approx}{B} = C = -3\nu$),

$$-\frac{d\underline{q}}{dt} + \underset{\approx}{B} \underline{q} = 0 , \quad 5.2.13$$

* We shall use the symbols (\sim) and $(^\dagger)$ to indicate the transpose and Hermitian conjugate of a matrix.

$$-\frac{d}{dt} \underset{\approx}{\Lambda} + \underset{\approx}{\Lambda} \underset{\approx}{B} + (\underset{\approx}{\Lambda} \underset{\approx}{B}) = 2\underset{\approx}{A} ; \quad 5.2.14$$

and to satisfy the initial conditions 5.2.8 we must have

$$K(t) \Big|_{t=0} = 0 ; \quad \underset{\approx}{\Lambda}(t) \Big|_{t=0} = 0 ; \quad \text{and} \quad \underset{\approx}{q}(t) \Big|_{t=0} = \underset{\approx}{w}_0 . \quad 5.2.15$$

A solution of the form 5.2.10 implies that $g(\underset{\approx}{w}, t; \underset{\approx}{v}_0)$ at any one time has a Gaussian probability distribution in phase space ($\underset{\approx}{w}$ space) with mean $\underset{\approx}{q}(t)$ and covariance matrix $\underset{\approx}{\Lambda}(t)$. Physically $g(\underset{\approx}{w}, t; \underset{\approx}{v}_0)$ can be interpreted as the probabilistic description of the random walk motion of a particle in phase space knowing it was at $\underset{\approx}{x} = 0$ and $\underset{\approx}{v} = \underset{\approx}{v}_0$ at $t = 0$. The 6-dimensional vector $\underset{\approx}{q}(t)$ with three-dimensional components $\underset{\approx}{q}^x(t)$ and $\underset{\approx}{q}^v(t)$ (i.e., we define $\underset{\approx}{q}(t) = \begin{bmatrix} \underset{\approx}{q}^x(t) \\ \underset{\approx}{q}^v(t) \end{bmatrix}$) gives us the expected position in phase space. $\underset{\approx}{q}^x(t)$ is then the expected trajectory of the particle in physical space and $\underset{\approx}{q}^v(t)$ the expected velocity at any one time. We can write $\underset{\approx}{\Lambda}$ in terms of its 3×3 submatrices $\underset{\approx}{\Lambda}^{xx}(t)$, $\underset{\approx}{\Lambda}^{xv}(t)$, $\underset{\approx}{\Lambda}^{vx}(t)$, $\underset{\approx}{\Lambda}^{vv}(t)$, then $\underset{\approx}{\Lambda}^{xx}(t)$ is a measure of the uncertainty or the amount of probabilistic diffusion of the position of the particle in physical space and $\underset{\approx}{\Lambda}^{vv}(t)$ the corresponding one in velocity space.

From 5.2.12 and the initial condition for $K(t)$ we get: $K(t) = 0$; which means that $g(\underset{\approx}{w}, t; \underset{\approx}{v}_0)$ normalizes to unity at all times.

Equation 5.2.13 represents Newton's equations for the mean position and velocity of the particle and can readily be solved to give:

$$\underset{\approx}{q}^v(t) = \underset{\approx}{M}(t) \cdot \underset{\approx}{v}_0 \quad 5.2.16$$

and

$$\tilde{\mathbf{q}}^{\mathbf{x}}(t) = \tilde{\mathbf{L}}(t) \cdot \tilde{\mathbf{v}}_0, \quad 5.2.17$$

where $\tilde{\mathbf{M}}(t)$ and $\tilde{\mathbf{L}}(t)$ are 3×3 matrices:

$$\tilde{\mathbf{M}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega t & -\sin \Omega t \\ 0 & \sin \Omega t & \cos \Omega t \end{bmatrix} e^{-\nu t} \quad 5.2.18$$

and

$$\tilde{\mathbf{L}}(t) = \int_0^t \tilde{\mathbf{M}}(t) dt = \begin{bmatrix} \frac{1-e^{-\nu t}}{\nu} & 0 & 0 \\ 0 & \frac{e^{-\nu t} \sin(\Omega t - \chi) + \sin \chi}{\sqrt{\nu^2 + \Omega^2}} & \frac{e^{-\nu t} \cos(\Omega t - \chi) - \cos \chi}{\sqrt{\nu^2 + \Omega^2}} \\ 0 & \frac{-e^{-\nu t} \cos(\Omega t - \chi) + \cos \chi}{\sqrt{\nu^2 + \Omega^2}} & \frac{e^{-\nu t} \sin(\Omega t - \chi) + \sin \chi}{\sqrt{\nu^2 + \Omega^2}} \end{bmatrix}.$$

Here, $\chi = \tan^{-1}(\nu/\Omega)$.

To solve 5.2.14 it is convenient to transform it to a representation in which $\tilde{\mathbf{B}}^{\nu\nu}$ is diagonal. This we achieve with a unitary transformation matrix (6×6)

$$\tilde{\mathbf{T}} = \begin{bmatrix} \tilde{\mathbf{U}} & 0 \\ 0 & \tilde{\mathbf{U}} \end{bmatrix}, \quad 5.2.19$$

where $\tilde{\mathbf{U}}$ is in itself a 3×3 unitary (i.e., $\tilde{\mathbf{U}}^{-1} = \tilde{\mathbf{U}}^\dagger$) defined by

$$\underset{\approx}{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & -i & 1 \\ 0 & i & 1 \end{bmatrix} . \quad 5.2.20$$

If we let $\underset{\approx}{\Lambda}' = \underset{\approx}{T} \underset{\approx}{\Lambda} \underset{\approx}{T}^{-1}$, $\underset{\approx}{B}' = \underset{\approx}{T} \underset{\approx}{B} \underset{\approx}{T}^{-1}$, and $\underset{\approx}{A}' = \underset{\approx}{T} \underset{\approx}{A} \underset{\approx}{T}^{-1}$, the transformed equation for $\underset{\approx}{\Lambda}'$ is then

$$-\frac{d\underset{\approx}{\Lambda}'}{dt} + \underset{\approx}{\Lambda}' \underset{\approx}{B}' + (\underset{\approx}{\Lambda}' \underset{\approx}{B}')^\dagger = 2\underset{\approx}{A}' . \quad 5.2.21$$

Any 3×3 submatrix of either $\underset{\approx}{\Lambda}, \underset{\approx}{B}$ or $\underset{\approx}{A}$ transforms independently of the other submatrices; for instance, $\underset{\approx}{B}^{xx}$ transforms to $\underset{\approx}{B}'^{xx} = \underset{\approx}{U} \underset{\approx}{B}^{xx} \underset{\approx}{U}^{-1}$, similarly $\underset{\approx}{\Lambda}'^{vv} = \underset{\approx}{U} \underset{\approx}{\Lambda}^{vv} \underset{\approx}{U}^{-1}$, etc. The unit matrix $\underset{\approx}{I}$ is left invariant; so, $\underset{\approx}{A}'$ is still

$$\underset{\approx}{A}' = \begin{bmatrix} 0 & | & 0 \\ \hline 0 & | & -v u^2 \underset{\approx}{I} \end{bmatrix} \quad 5.2.22a$$

and

$$\underset{\approx}{B}' = \begin{bmatrix} 0 & | & 0 \\ \hline \underset{\approx}{I} & | & \underset{\approx}{B}'^{vv} \end{bmatrix} , \quad 5.2.22b$$

with a diagonal $\underset{\approx}{B}'^{vv}$ as follows:

$$\underset{\approx}{B}'^{vv} = \begin{bmatrix} v & 0 & 0 \\ 0 & v - i\Omega & 0 \\ 0 & 0 & v + i\Omega \end{bmatrix} . \quad 5.2.23$$

We also note that our assumption of symmetry for $\underset{\approx}{\Lambda}$ does not contradict

5.2.14 since $\underset{\approx}{A}$ and $\underset{\approx}{\Lambda} \underset{\approx}{B} + (\underset{\approx}{\Lambda} \underset{\approx}{B})$ are symmetric. Furthermore, $\underset{\approx}{\Lambda}$ must be real since $\underset{\approx}{A}$ and $\underset{\approx}{B}$ are real. A symmetric and real matrix is Hermitian. Hermitian matrices

transform into Hermitian matrices under unitary transformation. Therefore, Λ'_{\approx} is also Hermitian, so $\Lambda'^{vv}_{\approx} = \Lambda'^{vv\dagger}_{\approx}$, $\Lambda'^{xx}_{\approx} = \Lambda'^{xx\dagger}_{\approx}$, and $\Lambda'^{xv}_{\approx} = \Lambda'^{xv\dagger}_{\approx}$. Using these properties of Λ'_{\approx} and the fact that B'^{vv}_{\approx} is diagonal we get the following submatrix equations

$$\frac{d\Lambda'^{vv}_{\approx}}{dt} - \Lambda'^{vv}_{\approx} (B'^{vv}_{\approx} + B'^{vv\dagger}_{\approx}) = 2\nu u^2 I_{\approx} \quad 5.2.24$$

$$\frac{d\Lambda'^{xv}_{\approx}}{dt} - \Lambda'^{xv}_{\approx} B'^{vv}_{\approx} = \Lambda'^{vv}_{\approx} \quad 5.2.25$$

$$\frac{d\Lambda'^{xx}_{\approx}}{dt} = \Lambda'^{xv}_{\approx} + \Lambda'^{xv\dagger}_{\approx}, \quad 5.2.26$$

which one can readily solve in that order. The fourth submatrix Λ'^{vx}_{\approx} can be obtained from $\Lambda'^{vx}_{\approx} = \Lambda'^{xv\dagger}_{\approx}$. The solution of 5.2.24 is given by

$$\Lambda'_{ij}{}^{vv} = +u^2(1 - e^{-2\nu t}) \delta_{ij} . \quad 5.2.27$$

Only the diagonal terms of Λ'^{vv}_{\approx} are non-zero, since the non-diagonal terms satisfy a homogeneous equation with homogeneous initial conditions. They are furthermore equal since $B'^{vv}_{\approx} + B'^{vv\dagger}_{\approx} = 2\nu I_{\approx}$ and therefore they must satisfy the same equation.

The submatrix Λ'^{xv}_{\approx} is also diagonal for the same reasons as Λ'^{vv}_{\approx} but the terms are not equal since it involves B'^{vv}_{\approx} with different diagonal terms. Solving equation 5.2.25 for the diagonal terms one gets

$$\Lambda_{11}^{XV} = \frac{u^2}{v} \left[1 - e^{-vt} \right] e^{-vt} \quad 5.2.28$$

and

$$\Lambda_{22}^{XV} = \left(\Lambda_{33}^{XV} \right)^* = u^2 \left[\frac{1 - e^{-(v-i\Omega)t}}{v-i\Omega} + \frac{e^{-2vt} - e^{-(v+i\Omega)t}}{v+i\Omega} \right]. \quad 5.2.29$$

Similarly, the terms of Λ_{\approx}^{XX} are also diagonal and are obtained by direct integration of twice the real part of 5.2.29. They are given by

$$\Lambda_{11}^{XX} = \frac{2u^2}{v^2} \left[vt + 2(e^{-vt} - 1) + \frac{1 - e^{-2vt}}{2} \right] \quad 5.2.30$$

and

$$\Lambda_{22}^{XX} = \Lambda_{33}^{XX} = \frac{2u^2}{v^2 + \Omega^2} \left[vt - 2e^{-vt} \sin \chi t \sin(\Omega t - \chi) - 2 \sin^2 \chi + \frac{1 - e^{-2vt}}{2} \right]. \quad 5.2.31$$

Expressions 5.2.27 to 5.2.31 completely define Λ'_{\approx} . From them one can obtain Λ_{\approx} by the inverse transformation $\Lambda_{\approx} = T^{-1} \Lambda'_{\approx} T$. One finds that:

$$\Lambda_{\approx}^{XX} = \Lambda_{\approx}^{XX} \quad 5.2.32$$

$$\Lambda_{\approx}^{VV} = \Lambda_{\approx}^{VV} \quad 5.2.33$$

and the components of $\Lambda_{\approx}^{XV} = \Lambda_{\approx}^{VX}$ are given by:

$$\Lambda_{11}^{XV} = \Lambda_{11}^{XV} \quad 5.2.34$$

$$\Lambda_{22}^{XV} = \Lambda_{33}^{XV} = \text{Re } \Lambda_{22}^{XV} = \frac{u^2}{\sqrt{v^2 + \Omega^2}} \left[1 + e^{-2vt} - 2e^{-vt} \cos \Omega t \right] \sin \chi \quad 5.2.35$$

$$\Lambda_{23}^{XV} = -\Lambda_{32}^{XV} = \text{Im } \Lambda_{22}^{XV} = \frac{u^2}{\sqrt{v^2 + \Omega^2}} \left[(1 - e^{-2vt}) \cos \chi - 2 \sin \chi e^{-vt} \sin \Omega t \right] \quad 5.2.36$$

As it has been said before $G(\alpha, t; v_0)$ gives us all the information we need,

in any case we can formally write its inverse transform (see Cramer²⁸, Chapter 24),

$$g(\underline{w}, \underline{v}_o; t) = \frac{1}{(2\pi)^3 |\underline{\Lambda}|} \exp \left[-\frac{1}{2} (\underline{w} - \underline{q})^\dagger \cdot \underline{\Lambda}^{-1} \cdot (\underline{w} - \underline{q}) \right] . \quad 5.2.37$$

3. Evaluation of $\hat{I}(\underline{\xi}, z)$ and $\hat{S}(\underline{\xi}, z)$.

Inserting the expression for $\hat{g}(\underline{\xi}, \underline{v}, \underline{v}_o, z)$ and $\bar{J}(\underline{\xi}, \underline{v}_o)$ as given by 5.1.9 and 5.1.11 in the defining equation for $\hat{I}(\underline{\xi}, z)$ and making use of 5.2.9 we can write

$$\hat{I}(\underline{\xi}, z) = \int_0^\infty dt e^{-izt} \bar{I}(\underline{\xi}, t) , \quad 5.3.1$$

where $\bar{I}(\underline{\xi}, t)$, clearly the Laplace inverse of $\hat{I}(\underline{\xi}, z)$, is given by

$$\bar{I}(\underline{\xi}, t) = H(\underline{\xi}) \int d\underline{v}_o G(\underline{\alpha}, t; \underline{v}_o) \Big|_{\eta=0} \varphi(\underline{v}_o) . \quad 5.3.2$$

But,

$$G(\underline{\alpha}, t; \underline{v}_o) \Big|_{\eta=0} = \exp \left(-i\tilde{\alpha} \cdot \underline{q} - \frac{1}{2} \tilde{\alpha} \cdot \underline{\Lambda} \cdot \underline{\alpha} \right) \Big|_{\eta=0} = \exp \left(-i\tilde{\xi} \cdot \underline{L}(t) \cdot \underline{v}_o - \frac{1}{2} \tilde{\xi} \cdot \underline{\Lambda}^{xx}(t) \cdot \underline{\xi} \right) \quad 5.3.3$$

so:

$$\bar{I}(\underline{\xi}, t) = H(\underline{\xi}) \exp \left(-\frac{1}{2} \tilde{\xi} \cdot \underline{\Lambda}^{xx}(t) \cdot \underline{\xi} \right) \int d\underline{v}_o \frac{1}{(2\pi u_o^2)^{3/2}} \exp \left(-\frac{v_o^2}{2u^2} - i\tilde{\xi} \cdot \underline{L}(t) \cdot \underline{v}_o \right) . \quad 5.3.4$$

The integral above can be readily evaluated by completing the squares to give

$$\bar{I}(\underline{\xi}, t) = H(\underline{\xi}) \exp \left(-\frac{u^2}{2} \tilde{\xi} \cdot \underline{\Lambda}^{xx}(t) \cdot \underline{\xi} - \frac{1}{2} \tilde{\xi} \cdot \underline{L}(t) \cdot \underline{\tilde{L}}(t) \cdot \underline{\xi} \right) . \quad 5.3.5$$

Using the expressions obtained for $\Lambda_{\approx}^{\text{xx}}(t)$ and $L(t)$ in 5.1.12, 5.2.33, and

5.2.34 we get

$$\bar{I}(\underline{\xi}, t) = \bar{H}(\underline{\xi}) \exp \left[-\frac{\xi_{\parallel}^2 u^2}{v^2} \left(vt - 1 + e^{-vt} \right) - \frac{\xi_{\perp}^2 u^2}{v^2 + \Omega^2} \left(\cos 2\chi + vt - e^{-vt} \cos(\Omega t - 2\chi) \right) \right], \quad 5.3.6$$

where $\xi_{\parallel}^2 = \xi_1^2$ and $\xi_{\perp}^2 = \xi_2^2 + \xi_3^2$. Similarly, we can write $\hat{S}(\underline{\xi}, t)$ defined in 5.1.17

$$\hat{S}(\underline{\xi}, z) = \int_0^{\infty} dt e^{-izt} \bar{S}(\underline{\xi}, t), \quad 5.3.7$$

where $\bar{S}(\underline{\xi}, t)$, the Laplace inverse of $\hat{S}(\underline{\xi}, z)$, is given by

$$\bar{S}(\underline{\xi}, t) = -u^2 \int d\underline{v}_o G(\underline{\alpha}, t; \underline{v}_o) \Big|_{\eta=0} \underline{\tilde{\xi}} \cdot \frac{\partial \varphi(\underline{v}_o)}{\partial \underline{v}_o}, \quad 5.3.7$$

or

$$\bar{S}(\underline{\xi}, t) = -u^2 \int d\underline{v}_o i \underline{\tilde{\xi}} \cdot \frac{\partial \varphi(\underline{v}_o)}{\partial \underline{v}_o} \exp \left(-i \underline{\tilde{\xi}} \cdot \underline{L}(t) \cdot \underline{v}_o - \frac{1}{2} \underline{\tilde{\xi}} \cdot \Lambda_{\approx}^{\text{xx}}(t) \cdot \underline{\xi} \right). \quad 5.3.8$$

Integrating 5.3.8 by parts we get

$$\bar{S}(\underline{\xi}, t) = u^2 \underline{\tilde{\xi}} \cdot \underline{L}(t) \cdot \underline{\xi} \exp \left(-\frac{1}{2} \underline{\tilde{\xi}} \cdot \Lambda_{\approx}^{\text{xx}}(t) \cdot \underline{\xi} \right) \int d\underline{v}_o \frac{1}{(2\pi u^2)^{3/2}} \exp \left(-\frac{v_o^2}{2u^2} - i \underline{\tilde{\xi}} \cdot \underline{L}(t) \cdot \underline{v}_o \right). \quad 5.3.9$$

The integral in this expression is the same as the one in 5.3.3. Evaluating the integral and writing explicitly the quadratic expressions in $\underline{\xi}$, one gets

$$\begin{aligned} \bar{S}(\underline{\xi}, t) = & \left\{ \frac{\xi_{\perp}^2 u^2}{\sqrt{v^2 + \Omega^2}} \left[\sin \chi + e^{-vt} \sin(\Omega t - \chi) \right] + \frac{\xi_{\parallel}^2 u^2}{v} \left(1 - e^{-vt} \right) \right\} \\ & \cdot \exp \left\{ -\frac{\xi_{\parallel}^2 u^2}{v^2} \left(vt - 1 + e^{-vt} \right) - \frac{\xi_{\perp}^2 u^2}{v^2 + \Omega^2} \left[\cos 2\chi + vt - e^{-vt} \cos(\Omega t - 2\chi) \right] \right\}. \end{aligned} \quad 5.3.10$$

It is interesting to notice that

$$\bar{S}(\xi, t) = -\frac{1}{\bar{H}(\xi)} \frac{d}{dt} \bar{I}(\xi, t) \quad 5.3.11$$

and consequently

$$\hat{S}(\xi, z) = \frac{1}{\bar{H}(\xi)} \left(1 - iz \hat{I}(\xi, z) \right) \quad 5.3.12$$

a property that does not become apparent until $\bar{S}(\xi, t)$ and $\bar{I}(\xi, t)$ are fully evaluated.

Inserting 5.3.6 and 5.3.10 into 5.3.1 and 5.3.7 one gets the final expression for $\hat{I}(\xi, z)$ and $\hat{S}(\xi, z)$.

One should consider these expressions as one of many possible integral representations. For instance, in the case of no collisions and no magnetic field ($\nu = 0$, $\Omega = 0$) they take the form

$$\hat{I}(\xi, z) = \bar{H}(\xi) \int_0^\infty dt e^{-izt} e^{-\xi^2 u^2 t^2 / 2}, \quad 5.3.13$$

and

$$\hat{S}(\xi, z) = \int_0^\infty dt e^{-izt} \xi^2 u^2 t e^{-\xi^2 u^2 t^2 / 2}. \quad 5.3.14$$

Had we performed the t -integration before the \underline{v}_0 integration in their definitions we would have obtained the following expressions

$$\hat{I}(\xi, z) = \bar{H}(\xi) \int_{-\infty}^\infty \frac{\varphi(\underline{v}_0)}{i\xi \cdot \underline{v}_0 + iz} d\underline{v}_0 \quad 5.3.15$$

$$\frac{1}{\xi^2 h^2} \hat{S}(\xi, z) = -\frac{\omega_\mu^2}{\xi^2} i\xi \cdot \int_{-\infty}^\infty \frac{\varphi'(\underline{v}_0)}{i\xi \cdot \underline{v}_0 + iz} d\underline{v}_0, \quad 5.3.16$$

which are more common representations in the plasma physics literature.

Being able to express $\hat{I}(\xi, z)$, $\hat{S}(\xi, z)$ in an integral representation which is essentially the Laplace transform of an ordinary function of time plays an important part in the numerical evaluation technique described in Chapter VI.

4. On the Time Behaviour of $\bar{I}(\underline{k}, t)$ and $\bar{S}(\underline{k}, t)$ — Physical Interpretation.

In order to understand and give some physical interpretation to the numerical results presented in Chapter VI it is appropriate that we discuss the behaviour of the functions $\bar{I}(\underline{k}, t)$ and $\bar{S}(\underline{k}, t)$ as a function of time. We are interested in particular in the first one since we shall see that most of the features in $\bar{n}_1^e(\underline{k}, t)$ can be explained in terms of qualitatively similar ones in $\bar{I}^i(\underline{k}, t)$ (i stands for the ion components).

Let us first note that $\bar{I}(\underline{k}, t) e^{-i\underline{k} \cdot \underline{x}}$ can be thought of as being the solution of

$$\mathcal{L}h_{\underline{k}}(\underline{x}, t) = 0 \quad 5.4.1$$

with initial condition

$$h_{\underline{k}}(\underline{x}, t) \Big|_{t=0} = \bar{H}(\underline{k}) e^{-i\underline{k} \cdot \underline{x}} \varphi(\underline{v}) \quad , \quad 5.4.2$$

since in terms of the Green's function of \mathcal{L} one can write the solution $h_{\underline{k}}(\underline{x}, t)$ as

$$h_{\underline{k}}(\underline{x}, t) = \int d\underline{v} d\underline{v}_0 d\underline{x}_0 g(\underline{x} - \underline{x}_0, \underline{v}, \underline{v}_0, t) \bar{H}(\underline{k}) e^{-i\underline{k} \cdot \underline{x}_0} \varphi(\underline{v}_0)$$

or

$$h_{\underline{k}}(\underline{x}, t) = \bar{I}(\underline{k}, t) e^{-i\underline{k} \cdot \underline{x}} \quad 5.4.3$$

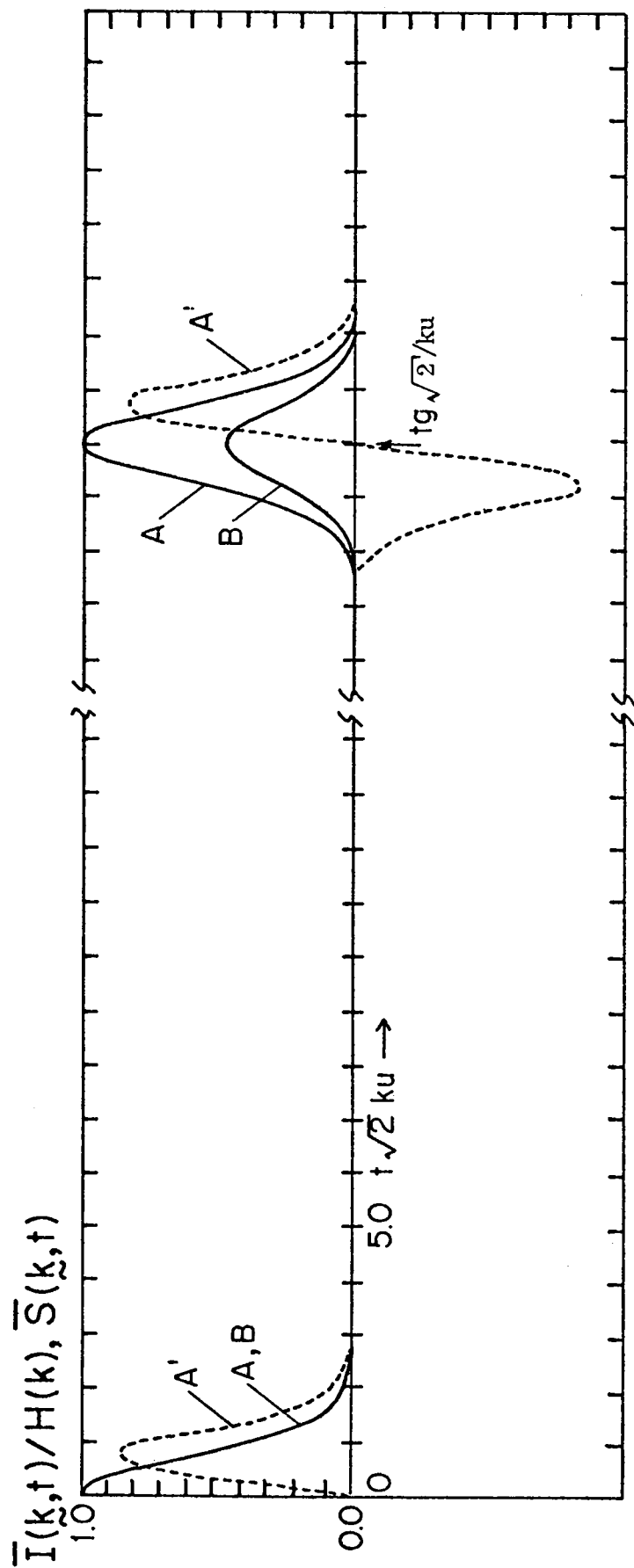


FIG. 3. Sketch of the time behaviour of the functions $\bar{I}(\tilde{k}, t)/H(\tilde{k})$ (solid lines) and $\bar{S}(\tilde{k}, t)$ (dotted lines) in the case $\Omega/\tilde{k}u \ll 1$. Curves A and A' correspond to $\alpha = 0^\circ$ and $\nu = 0$, curve B shows the effect of either a small α or small ν .

by definition of $\bar{I}(\underline{k}, t) e^{-i \underline{k} \cdot \underline{x}}$.

Physically then we can interpret $\bar{I}(\underline{k}, t)$ as the amplitude (as a function of time) of a wave, with wave vector \underline{k} , in a gas without collective Coulomb interactions (the collective Coulomb interaction term is not included in \mathcal{L}) which has initially been set with amplitude $\bar{H}(\underline{k})$ and Maxwellian distribution $\varphi(\underline{v})$. The solution 5.4.3 written in its integral form can be interpreted as the density produced by the superposition of the average density produced by single particles in random walk in a magnetic field (physical interpretation of the Green's function g) which were initially at \underline{x}_0 with velocity \underline{v}_0 with a distribution $\bar{H}(\underline{k}) e^{-i \underline{k} \cdot \underline{x}}$ in physical space and a velocity distribution $\varphi(\underline{v}_0)$. Such an interpretation will allow us to explain the behaviour of $\bar{I}(\underline{k}, t)$ (and of $\bar{n}_1^e(\underline{k}, t)$ later on) in terms of the motions of individual particles in a magnetic field.

We shall consider two cases depending on whether the wavelength (divided by 2π) is smaller or larger than the gyro-radius or equivalently whether Ω/kv is smaller or larger than unity.

Figure 3 shows schematically a typical plot of $\bar{I}(\underline{k}, t)$ in the case $\Omega/kv \ll 1$. This could be taken as a typical $\bar{I}(\underline{k}, t)$ for the ions under typical ionospheric conditions for small angles $\alpha = 90^\circ - \theta$, (the complement of the angle θ between \underline{k} and \underline{B}), and without much collisional effect.

The shape of the curve and the effects of the angle α and collision frequency ν can be better understood if we consider first the case where $\alpha = 0$ (\underline{k} perpendicular to \underline{B}) and $\nu = 0$. In terms of a non-dimensional time $t' = kv t / \sqrt{2}$ and non-dimen-

sional frequency $\Omega' = \sqrt{2} \Omega / ku$, we can write 5.3.6 in this case as:

$$\bar{I}(\underline{k}, t) = H(\underline{k}) \exp \left[-2/\Omega'^2 (1 - \cos \Omega' t') \right] . \quad 5.4.4$$

We are assuming now that $\Omega / ku \ll 1$ or, that $\Omega' \ll 1$, therefore, $2/\Omega'^2$ is a large factor and $\bar{I}(\underline{k}, t)$ has significant values only for those values of t' which make $(1 - \cos \Omega' t')$ small. These correspond to values of $\Omega' t'$ not far from $0, 2\pi, 4\pi$, etc. An expansion of $(1 - \cos \Omega' t')$ around those values of t' gives us $(1 - \cos \Omega' t') \sim \Omega'^2 / 2 \left(t' - N(2\pi/\Omega') \right)^2 - \Omega'^2 / 4 \left(t' - N(2\pi/\Omega') \right)^4 \dots$ where N is an integer $N=0, 1, 2, 3$, etc. which depends on the particular multiple of 2π we are expanding around. We can therefore approximate $\bar{I}(\underline{k}, t)$ for the sake of this discussion as

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) \sum_{N=0}^{\infty} e^{-(t' - N 2\pi/\Omega')^2} , \quad 5.4.5$$

which correspond to gaussians centered around $t=0, 2\pi, 4\pi$, etc.

If there were no magnetic field the function $\bar{I}(\underline{k}, t)$ would be given by (no approximations):

$$\bar{I}(\underline{k}, t) = \bar{H}(\underline{k}) e^{-t'^2}$$

and the wave would have negligible amplitude in about the time it takes a thermal particle (with velocity μ) to travel half a wavelength, i.e., in a time $t' = \pi/\sqrt{2}$. With the presence of the magnetic field the behaviour of $\bar{I}(\underline{k}, t)$ does not change much for values of t' of order unity, which is to be expected, since under the assumption $ku \ll \Omega$ the trajectories of the particles for those short times will differ

slightly from the straight lines in the non-magnetic field case. But at a time equal to a gyro-period $t'_g = 2\pi/\Omega'$, the wave will grow again to the same amplitude it had originally. This is explained by the fact that each particle, regardless of its initial velocity or position, will come back after a gyro-period to the same projected position (projected on a plane perpendicular to \underline{B} and containing \underline{k}) it had at $t=0$, and therefore, reconstructing whatever wave was initially there before. Any motion along \underline{B} , and therefore perpendicular to \underline{k} , does not affect the amplitude of the wave.

For values of α different from zero, $\bar{I}(\underline{k}, t)$ takes the form:

$$\bar{I}(\underline{k}, t) = \bar{H}(\underline{k}) \exp \left[-t'^2 \sin^2 \alpha - 2 \cos^2 \alpha / \Omega'^2 (1 - \cos \Omega' t') \right], \quad 5.4.6$$

which we can split in two factors, a periodic one as before due to $(1 - \cos \Omega' t')$, which we can approximate in similar fashion, and an $e^{-t'^2 \sin^2 \alpha}$ factor, so

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) e^{-t'^2 \sin^2 \alpha} \sum_{N=0}^{\infty} e^{-\cos^2 \alpha (t' - N 2\pi/\Omega')^2}. \quad 5.4.7$$

Because of the $e^{-t'^2 \sin^2 \alpha}$ factor the amplitude of the wave at the gyro-periods is not as large as before. At $t' = t'_g = 2\pi/\Omega'$ it will be $\bar{H}(\underline{k}) e^{-(2\pi/\Omega')^2 \sin^2 \alpha}$. For sufficiently small Ω' , a relative small angle α is sufficient to reduce the amplitude of the "gyro-peaks" to practically zero amplitude. For a magnetic field of 0.3 gauss, a temperature of 1000°K, an oxygen ion (typical ionospheric parameters) and a wavelength of 1.5 meters, this critical angle is of the order of 3° . Physically, this attenuation is due to the diffusion of the particles along the magnetic field lines,

which are no longer perpendicular to \underline{k} , therefore, after a gyro-period a thermal particle would be displaced from its original position a projected distance along \underline{k} of the order of $(2\pi/\Omega') \sin \alpha/k$. If this distance is of the order of half a wavelength or larger there is no reconstruction of the wave.

By inspection of 5.3.6 we see that if the collision frequency ν is of the order Ω or larger (mean free path of the order of gyro-radius or smaller), after a gyro-period $t = 2\pi/\Omega$, the term responsible for any possible periodicity $e^{-\nu t} \cos(\Omega t - 2\chi)$ would be attenuated by the factor $e^{-\nu t}$ which for $t = 2\pi/\Omega$ would be at most $e^{-2\pi}$, a very small number. That there is no periodicity for such values of ν is by no means surprising, since then, one would hardly expect the individual particles to return to their original projected positions regardless of the angle α . But $\nu \geq \Omega$ is not the necessary condition for the (practically) complete attenuation of the "gyro-peaks." In fact for the case $\Omega/ku \ll 1$. We are considering now a critical value of ν occurs at a much lower value with respect to Ω .

Let us consider a case where $\nu \ll \Omega$, so that $e^{-\nu t} = 1 - \nu t$ could be considered a good approximation for times $t \leq t_g$. Let us also consider the more favorable (as far as amplitudes of the "gyro-peaks") and simpler case, where $\alpha = 0$. Under these conditions we can approximate $\bar{I}(\underline{k}, t')$ by

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) \exp \left[-2/\Omega'^2 (1 - \cos \Omega' t' + \nu t' + \nu t' \cos \Omega' t') \right], \quad 5.4.8$$

and by expanding around $t' = 0, 2\pi, 4\pi$, etc.

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) e^{-4\nu' t' / \Omega'^2} \sum_{N=0}^{\infty} e^{-1/\Omega'^2 (t' - N 2\pi/\Omega')^2} . \quad 5.4.9$$

We see that the first "gyro-peak" is down by a factor $e^{-8\pi\nu'/\Omega'^3}$ or $e^{-4\pi(\nu/\Omega)(k^2 u^2/\Omega^2)}$ and no matter how small (ν/Ω) may be, there is always a k for a given Ω which would make that factor sufficiently small to appreciably reduce the amplitude of this peak to even negligible values. The critical value of ν would be clearly given by:

$$\nu_{\text{crit}} = \Omega/4\pi (\Omega/ku)^2 = 2\pi^2 t_g^{-1} (ku t_g)^{-2} . \quad 5.4.10$$

We can also interpret this attenuation of the first "gyro-peak" in terms of the motion of a single particle. Because of the random field (Coulomb-collisions) the particle is subject to, we have to describe it in a probabilistic manner. Since we are considering small values of ν as compared to Ω , we would expect that in a gyro-period the particle would have returned in the average to a point close to the one in the collisionless case, but with a relatively constant probability for it to be within a sphere of uncertainty. Certainly, if the radius of this sphere, regardless of how small it may be with respect to the gyro-radius, is larger than half a wavelength we can not expect any reconstruction of the wave. The exact expression for the probabilistic description of the particle is given by our Green's functions $g(\underline{x}, \underline{\nu}, \underline{\nu}_0, t)$ and the radius of the sphere of uncertainty by the square root of the variance Λ_{ij}^{xx} given by 5.2.30 and 5.2.31, which at $t=t_g$ and for the small values of $\nu t_g = \nu 2\pi/\Omega$ we are considering, would be approximately isotropic and approxi-

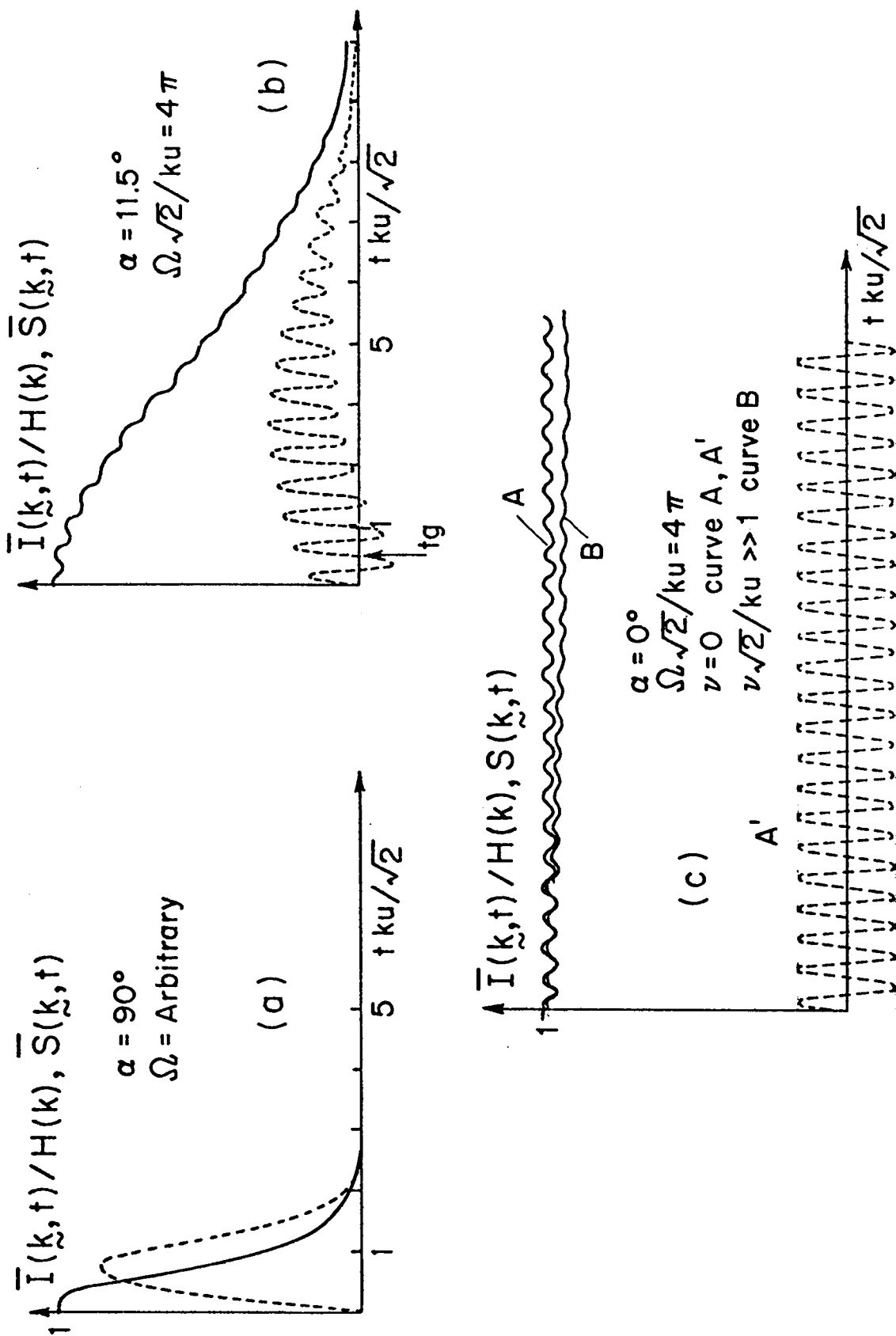


FIG. 4. Sketch of the time behaviour of the functions $\bar{I}(\underline{k}, t)/H(\underline{k})$ (solid lines) and $\bar{S}(\underline{k}, t)$ (dotted lines) in the case $\Omega/k u \gg 1$ for different values of α . Fig. 4c shows schematically the effect of collisions. Figures 4a and 4b are not affected much for values of $\nu \neq 0$ (provided $\nu \sqrt{2}/k u \ll 1$).

mately given by

$$\Lambda_{ij}^{xx} = \frac{2}{3} v t_g^3 u^2 \delta_{ij} , \quad 5.4.11$$

and if measured in wavelengths by

$$\frac{\Lambda_{ij}^{xx}}{\lambda^2} = \frac{2}{3} \frac{1}{4\pi^2} v t_g^3 k^2 u^2 , \quad 5.4.12$$

which gives us a critical frequency at $\Lambda_{ij}^{xx}/\lambda^2 = 1$ of the same form and approximately of the same magnitude as (5.4.10).

It is clear now that the actual criteria to decide whether to include the Fokker-Planck collision term (when ku/Ω is large) depend on the value of $v/v_{\text{crit}} = 4\pi(v/\Omega)$ ($k^2 u^2 / \Omega^2$). Notice that, whereas the magnitude of the moment-type collision term went down as v/ku as k increased, the effect of the Fokker-Planck term goes up as k^2 .

The second case to consider is when the reciprocal of the wave number $k^{-1} = 2\pi/\lambda$ is larger than the gyro-radius, i.e., when $\Omega/ku \gg 1$. ($\Omega' \gg 1$) Let us consider first the collisionless case. Under these conditions (using non-dimensional times and frequencies), we can approximate $\bar{I}(\underline{k}, t)$ in the form

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) \left(1 - \frac{1}{\Omega'^2} \cos^2 \alpha \right) e^{-t'^2 \sin^2 \alpha} + \frac{2}{\Omega'^2} \cos^2 \alpha e^{-t'^2 \sin^2 \alpha} \cos \Omega' t'. \quad 5.4.13$$

Figure 4 shows schematically the shape of this function for different values of α . It consists of a Gaussian like function of almost unit amplitude with a time constant of the order of $1/\sin \alpha$, and a small oscillating function with frequency Ω' and am-

plitude $1/\Omega'^2 \cos^2 \alpha$ attenuated in a Gaussian fashion with the same time constant. When $\alpha = \pi/2$, there is no oscillating component and $\bar{I}(\underline{k}, t)$ is of the same form as in the non-magnetic field case, and when $\alpha = 0$ the exponential term degenerates to a constant.

In terms of the motion of the individual particles we can interpret the behaviour of $\bar{I}(\underline{k}, t)$ for this case as being due to the diffusion of the particles, but this time constrained to move along the magnetic field lines with an effective velocity equal to the projection at the actual velocity of the center of gyration in the direction of the wave vector. The small oscillations are due to the small deviations of the particles from their centers of gyration (also projected in the direction of the wave vector).

For values of $v' \neq 0$ (but $v' \ll 1$ to be consistent with our assumptions for the validity of the collision model) we can obtain a somewhat simpler approximate expression for $\bar{I}(\underline{k}, t)$

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) \exp \left[-\frac{2 \sin^2 \alpha}{v'^2} \left(v' t' - 1 + e^{-v' t'} \right) \right] e^{-\frac{2 \cos^2 \alpha v' t'}{\Omega'^2} \left[\left(1 - \frac{\cos^2 \alpha}{\Omega'^2} \right) + \frac{2 \cos^2 \alpha}{\Omega'^2} e^{-v' t'} \cos \Omega' t' \right]} \quad . 5.4.14$$

The effect of collisions is not appreciable only for small angles α in a region where $\sin \alpha \leq v'$. For instance, at $\alpha = 0$ we can see that as compared with the collisionless case the amplitude of the small oscillations decay as $e^{-v' t'}$ and the "con-

stant" term decays (very slowly) as $e^{-(2 \cos^2 \alpha / \Omega'^2) v't'}$. For all other angles ($\sin \alpha \gg v'$) the expression goes to negligible value in a time of the order of $1/\sin \alpha$.

In this case we can write

$$\bar{I}(\underline{k}, t) \approx \bar{H}(\underline{k}) \exp \left[-t'^2 (1 - v't') \sin^2 \alpha \right] \cdot \left[\left(1 - \frac{2 \cos^2 \alpha}{\Omega'^2} \right) + \frac{2 \cos^2 \alpha}{\Omega'^2} (1 - v't') \cos \Omega't' \right] \quad 5.4.15$$

which differs slightly from the collisionless case, since before the terms involving $v't'$ have appreciable values the factor $e^{-\sin^2 \alpha t'^2}$ becomes very small.

For ionospheric applications and the wavelengths used only the $\bar{I}(\underline{k}, t)$ for the electrons belong to this second class ($\Omega/ku \gg 1$). But we shall see that in most cases of interest, with the exception of small values of $1/k^2 h^2$ (in which case $\bar{n}_1^e(\underline{k}, t) = \bar{I}^e(\underline{k}, t)$ to zeroth order in $1/k^2 h^2$) or small angles $\alpha \ll (m_e/m_i)^{1/2}$, the density $n_1^e(\underline{k}, t)$ bears little resemblance to the function $\bar{I}^e(\underline{k}, t)$.

We will not have any need for a physical interpretation of $\hat{S}(\underline{k}, t)$. Its behaviour for the different conditions discussed above can be obtained by simple differentiation of $\bar{I}(\underline{k}, t)$, since we have seen in 5.3.11 that $\bar{S}(\underline{k}, t) = -\left(1/\bar{H}(\underline{k})\right) \left(d\bar{I}(\underline{k}, t)/dt\right)$. In Figures 3 and 4 we show schematically some typical shapes of $\bar{S}(\underline{k}, t)$ which correspond to the curves for $\bar{I}(\underline{k}, t)$ depicted in the same figures.

5. Discussion of $\bar{n}_1^e(\underline{k}, t)$ and $\bar{n}_1^i(\underline{k}, t)$ in the $(m_e/m_i)^{1/2} = 0$ Approximation.

The analytical expression for $\bar{n}_1^e(\underline{k}, t)$ is given by the Laplace inverse of

5.1.21. It is a complicated expression to be discussed but numerically. Nevertheless something can be said about it, as well as $\bar{n}_1^e(\underline{k}, t)$, if we consider the case when $(m_e/m_i) = 0$. Since the real value of m_e/m_i is very small, whatever we can say about the $m_e/m_i = 0$ case can hopefully be generalized qualitatively to the real case and help us to understand and interpret the numerical results (using finite m_e/m_i) obtained in Chapter VI.

As far as the behaviour of $\bar{n}_1^e(\underline{k}, t)$ around plasma frequencies is concerned, taking $(m_e/m_i) = 0$ is a good approximation and it will not be discussed numerically.

Let us define a function

$$Y(t;) \equiv Y(t; \Omega, \nu, \alpha) = \exp \left[-\frac{\sin^2 \alpha}{\nu^2} (\nu t - 1 + e^{-\nu t}) - \frac{\cos^2 \alpha}{\nu^2 + \Omega^2} \left(\cos 2\chi + \nu t - e^{-\nu t} \cos(\Omega t - 2\chi) \right) \right], \quad 5.5.1$$

and its derivative

$$Y'(t;) \equiv Y'(t; \Omega, \nu, \alpha) = \frac{d}{dt} Y(t;) . \quad 5.5.2$$

Here, $\tan^{-1} \chi = \nu/\Omega$.

We also define their Laplace transforms

$$\hat{Y}(z;) \equiv \hat{Y}(z; \Omega, \nu, \alpha) = \int_0^\infty dt e^{-izt} Y(t; \Omega, \nu, \alpha) , \quad 5.5.3$$

and

$$\hat{Y}'(z;) \equiv \hat{Y}'(z; \Omega, \nu, \alpha) = \int_0^\infty dt e^{-izt} Y'(t; \Omega, \nu, \alpha) . \quad 5.5.4$$

In terms of these functions we can write

$$\bar{\Gamma}^{\mu}(\underline{k}, t) = \bar{H}^{\mu}(\underline{k}) Y(\underline{k}u_{\mu} t; \Omega_{\mu}/\underline{k}u_{\mu}, \nu_{\mu}/\underline{k}u_{\mu}, \alpha) \quad 5.5.5$$

$$\hat{\Gamma}^{\mu}(\underline{k}, t) = \bar{H}^{\mu}(\underline{k}) \hat{Y}(z/\underline{k}u_{\mu}; \Omega_{\mu}/\underline{k}u_{\mu}, \nu_{\mu}/\underline{k}u_{\mu}, \alpha), \quad 5.5.6$$

and using the relation

$$\bar{S}^{\mu}(\underline{k}, t) = -\frac{1}{H^{\mu}(\underline{k})} \frac{d}{dt} \bar{\Gamma}^{\mu}(\underline{k}, t) \quad 5.5.7$$

we have that

$$\bar{S}^{\mu}(\underline{k}, t) = -\underline{k}u_{\mu} Y'(\underline{k}u_{\mu} t; \Omega_{\mu}/\underline{k}u_{\mu}, \nu_{\mu}/\underline{k}u_{\mu}, \alpha) \quad 5.5.8$$

$$\hat{S}^{\mu}(\underline{k}, z) = \hat{Y}'(z/\underline{k}u_{\mu}; \Omega_{\mu}/\underline{k}u_{\mu}, \nu_{\mu}/\underline{k}u_{\mu}, \alpha) \quad 5.5.9$$

$$= 1 - iz/\underline{k}u_{\mu} \hat{Y}(z/\underline{k}u_{\mu}; \Omega_{\mu}/\underline{k}u_{\mu}, \nu_{\mu}/\underline{k}u_{\mu}, \alpha). \quad 5.5.10$$

We shall need the following properties of above functions

$$Y(t;) = 1 - \frac{t^2}{2!} + \nu \frac{t^3}{3!} + \left[3 + (\Omega^2 - \nu^2) \cos^2 \alpha \right] \frac{t^4}{4!} + 0(t^5) \quad 5.5.11$$

$$\hat{Y}(z;) \sim \frac{1}{iz} - \frac{1}{(iz)^3} + \nu \frac{1}{(iz)^4} + \left[3 + (\Omega^2 - \nu^2) \cos^2 \alpha \right] \frac{1}{(iz)^5} + 0(1/z^6) \quad 5.5.12$$

$$\hat{Y}'(z;) \sim \frac{1}{(iz)^2} - \nu \frac{1}{(iz)^3} + \left[3 + (\Omega^2 - \nu^2) \cos^2 \alpha \right] \frac{1}{(iz)^4} + 0(1/z^5) \quad 5.5.13$$

also

$$Y(z;) = m_0 - iz m_1 - z^2 \frac{m_2}{2!} + 0(z^3) \quad 5.5.14$$

and

$$\hat{Y}'(z;) = 1 - iz m_0 - z^2 \frac{m_1}{1!} + iz^3 \frac{m_2}{2!} + 0(z^4), \quad 5.5.15$$

where

$$m_n = \int_0^\infty t^n Y(t;) dt . \quad 5.5.16$$

Let us consider a single ion species i (with charge $Z_\mu = 1$). Using the Y functions defined above and in terms of the parameter $\delta = (m_e/m_i)^{1/2} = (ku_i/ku_e)$ we can write 5.1.21 in the form:

$$\begin{aligned} \hat{n}_1^i(\underline{k}, z) = & \frac{1}{ku_i} \bar{H}^i(k) \hat{Y}(z_i; i) + \frac{1}{ku_i} \\ & \frac{\kappa^2 \hat{Y}'(z_i; i) \left[\delta \bar{H}^e(k) \hat{Y}(\delta z_i; e) - \bar{H}^i(k) \hat{Y}(z_i; i) \right]}{1 + \kappa^2 \left[\hat{Y}'(z_i; i) + \hat{Y}(\delta z_i; e) \right]} , \end{aligned} \quad 5.5.17$$

where $z_i = z/ku_i$, $\kappa^2 = 1/k^2 h_e^2 = 1/k^2 h_i^2$, and $(; \mu)$ stands for $(; \Omega_\mu/ku_\mu, \nu/ku_\mu, \alpha)$.

If we let δ go to zero in 5.5.17, we obtain:

$$\begin{aligned} \hat{n}_1^i(\underline{k}, z) = & \frac{1}{ku_i} \bar{H}^i(k) \hat{Y}(z_i; i) - \frac{1}{ku_i} \\ & \frac{\kappa^2/(1+\kappa^2) \hat{Y}'(z_i; i) \bar{H}^i(k) \hat{Y}(z_i; i)}{1 + \kappa^2/(1+\kappa^2) \hat{Y}'(z_i; i)} , \end{aligned} \quad 5.5.18$$

or

$$\hat{n}_1^i(\underline{k}, z) = \frac{1}{ku_i} \frac{\bar{H}^i(k) \hat{Y}(z_i; i)}{1 + \kappa^2/(1+\kappa^2) \hat{Y}'(z_i; i)} , \quad 5.5.19$$

since $\hat{Y}(0;) = m_0$ and bounded (except when $\alpha = 0$ with $\nu = 0$ and $\Omega \neq 0$) and

$\hat{Y}(0;) = 1$.

In terms of the functions $\hat{I}^i(\underline{k}, z)$ and $\hat{S}^i(\underline{k}, z)$ we can write then, that for $\delta = 0$,

$$\hat{n}_1^i(\underline{k}, z) = \frac{1}{1 + \kappa^2/(1 + \kappa^2)} \hat{I}^i(\underline{k}, z) . \quad 5.5.20$$

We can use our general solution for $\hat{n}_1^\mu(\underline{k}, t)$ to obtain an expression for a hypothetical case of a single component plasma. The density $\hat{n}_1(\underline{k}, z)$ in such a case is given by

$$\hat{n}_1(\underline{k}, z) = \frac{1}{1 + \kappa^2 \hat{S}(\underline{k}, z)} \hat{I}(\underline{k}, z) . \quad 5.5.21$$

Comparing 5.5.21 with 5.5.20 we note they differ by the factor $1/(1 + \kappa^2)$ multiplying $\hat{S}(\underline{k}, z)$. From the definitions of $\hat{S}(\underline{k}, z)$ (5.1.17), we see that $\hat{S}(\underline{k}, z)$ is proportional to the charge squared of the particles e^2 . For the purpose of giving a physical interpretation to 5.5.20 we can associate the factor $1/(1 + \kappa^2)$ with e^2 . In most cases of interest the wavelength $2\pi/k$ is much larger than the Debye length, making κ^2 a very large number. Therefore, we can interpret 5.5.20 by saying that the ion gas in the presence of the electrons behaves as a single component plasma but with reduced charge $e/(1 + \kappa^2)^{1/2}$ (as far as collective Coulomb interactions go). This charge is much smaller than the actual one, so we can also say that the presence of the electrons effectively (almost) neutralizes the charge of the ions. Alternatively, we can associate $1/(1 + \kappa^2)$ with the Fourier transform of the Coulomb field $\hat{\phi}(\underline{k}) = \underline{k}/k^2$ in the definition of $\hat{S}(\underline{k}, z)$, and redefine an equivalent $\hat{\phi}'(\underline{k}) = \underline{k}/k^2 \cdot 1/(1 + \kappa^2) = (\underline{k}/k^2) \cdot (k^2 h_e^2 / (1 + k^2 h_e^2))$ which is the Fourier transform of $(1/4\pi)(\partial/\partial r)(e^{-r/h}/r)$. Thus, working from the

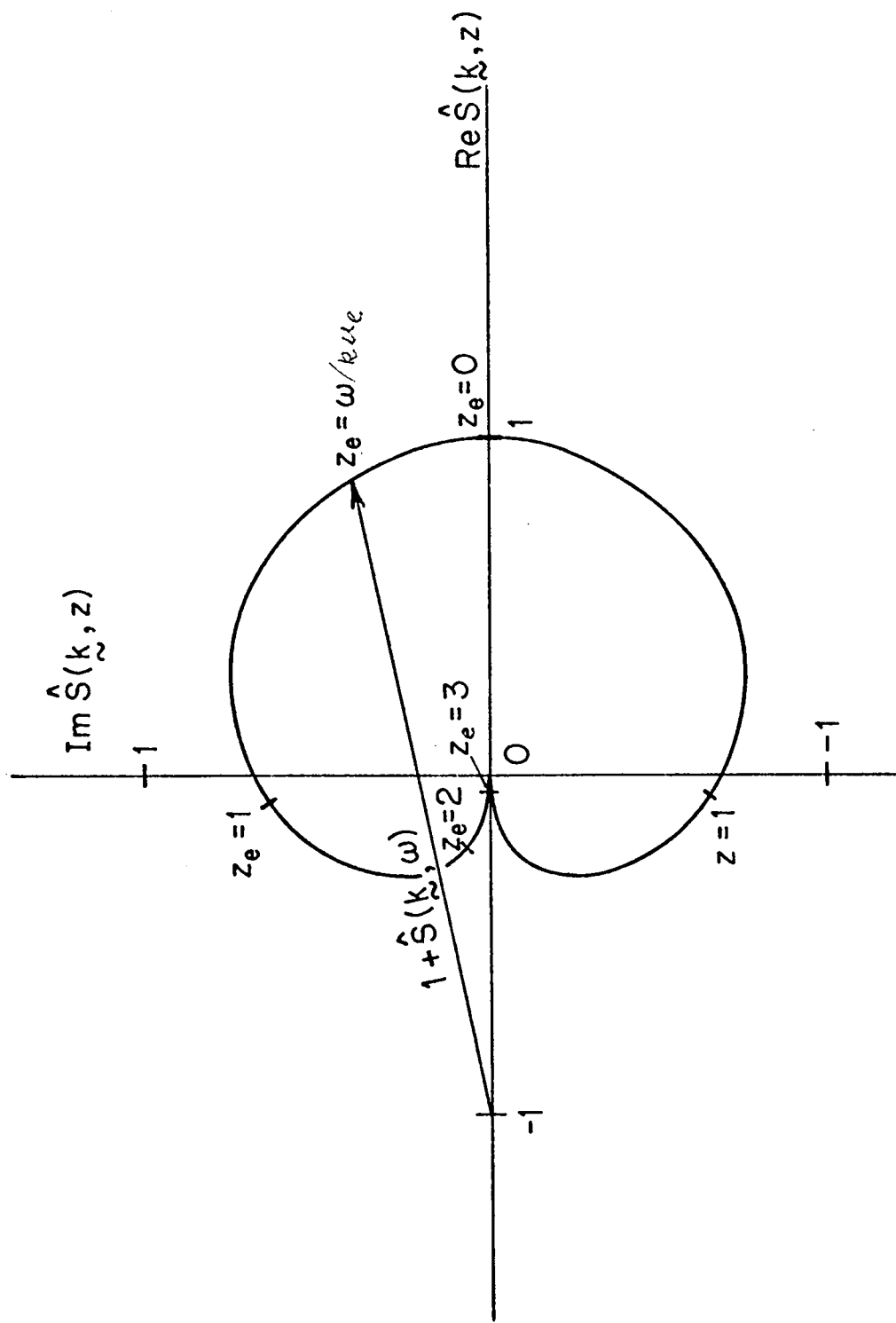


FIG. 5. Nyquist plot for the function $\hat{S}(k, z)$ in the case of no magnetic field showing the behaviour of $1 + \hat{S}(k, z)$ for real values of $z = \omega$. Here $z_e = z / k u_e$.

solution to the equation, we see that in the limit $m_e/m_i = 0$ the dynamics of the ions are given by a single plasma equation (decoupled from the electron one) but with effective interparticle potentials given by $e^{-r/h_e}/r$, i.e., the Debye shielded potential. Clearly for wavelengths larger than the shielding distance h_e the collective interactions are going to be greatly reduced. This shielded behaviour of the ions explains the almost direct relationship between $\bar{n}_1^i(\underline{k}, t)$ and $\bar{I}_1^i(\underline{k}, t)$, i.e., between the behaviour of the actual ion gas $\bar{n}_1^i(\underline{k}, t)$ and the behaviour of a hypothetical ion gas with no collective Coulomb interaction, that we shall find from our numerical calculations* and that we claimed in discussing our collision model. We can illustrate this without numerical computation, by considering the relatively simpler particular case in which $\Omega = 0$ and $\nu = 0$. In this case

$$\hat{S}^i(\underline{k}, z) = \int_0^\infty dt \, k u_i \, t e^{-t^2/2} e^{-izt} = -i \frac{1}{2} Z(z) \quad , \quad 5.5.22$$

where $Z(z)$ is the "plasma dispersion function" tabulated by Fried and Conte.²⁹

For large values of κ we can write $1 + \kappa^2/(1 + \kappa^2) \hat{S}^i(\underline{k}, z) \approx 1 + \hat{S}^i(\underline{k}, z)$. Figure 5 shows a Nyquist diagram for $\hat{S}^i(\underline{k}, z)$ from which we can obtain graphically $1 + \hat{S}^i(\underline{k}, z)$. The point to notice is that the absolute value of $1 + \hat{S}^i(\underline{k}, z)$ does not deviate more than 50% from a mean value. These deviations are by no means negligible but tell us that $\bar{n}_1^i(\underline{k}, t)$, the Laplace inverse of 5.5.20, is not going to deviate much from $\bar{I}_1^i(\underline{k}, t)$ the Laplace inverse of $\hat{I}_1^i(\underline{k}, z)$.

* Actually the numerical calculations are for $\bar{n}_1^e(\underline{k}, t)$ but we shall see soon a direct relationship between the two (in the $m_e/m_i = 0$ approximation).

Let us now consider the density of the electrons. Using the Y functions and the same frequency scale z_i as for the ions we can write $\hat{n}_1^e(\underline{k}, z)$ from 5.1.21 as

$$\hat{n}_1^e(\underline{k}, z) = \delta \frac{\bar{H}^e(\underline{k})}{ku_i} \hat{Y}(\delta z_i; e) + \frac{1}{ku_i} \frac{\kappa^2 Y'(\delta z_i; e) \left[\bar{H}^i(\underline{k}) \hat{Y}(z_i; i) - \delta \bar{H}^e(\underline{k}) \hat{Y}(\delta z_i; e) \right]}{1 + \kappa^2 \left[\hat{Y}(z_i; i) + \hat{Y}'(\delta z_i; e) \right]},$$

which for $\delta = 0$ becomes,

$$\hat{n}_1^e(\underline{k}, z) = \frac{1}{ku_i} \frac{\kappa^2}{1 + \kappa^2} \frac{\bar{H}^i(\underline{k}) \hat{Y}(z_i; i)}{1 + \kappa^2 / (1 + \kappa^2) \hat{Y}'(z_i; i)}. \quad 5.5.23$$

Since as before $Y(0;)$ is a constant and $Y'(0;) = 1$.

In terms of $\hat{n}_1^i(\underline{k}, z)$ as given by 5.5.19 we obtain that

$$\hat{n}_1^e(\underline{k}, z) = \frac{\kappa^2}{1 + \kappa^2} \hat{n}_1^i(\underline{k}, z), \quad 5.5.24$$

and obviously

$$\bar{n}_1^e(\underline{k}, t) = \frac{\kappa^2}{1 + \kappa^2} \bar{n}_1^i(\underline{k}, t), \quad 5.5.25$$

which for large κ becomes $\bar{n}_1^e(\underline{k}, t) \approx \bar{n}_1^i(\underline{k}, t)!$ This relationship together with the close relationship between $\bar{n}_1^i(\underline{k}, t)$ and $\bar{I}^i(\underline{k}, t)$ justifies the importance we have given to $\bar{I}^i(\underline{k}, t)$ and the motion of single non-interacting ions, even though our interest is in the electron density.

The expression 5.5.23 does not give a complete picture for $\hat{n}_1^e(\underline{k}, z)$ since in the process of going to the limit $\delta = 0$ and keeping ku_i finite we have indirectly sent all frequency detail of scale ku_e to infinity. In order to recover the high

frequency behaviour we write 5.5.17 in terms of a non-dimensional frequency $z_e = z/\kappa u_e$,

$$\hat{n}_1^e(\underline{k}, z) = \frac{\bar{H}^e(\underline{k})}{\kappa u_e} \hat{Y}(z_e; e) + \frac{1}{\kappa u_e} \frac{\kappa^2 \hat{Y}'(z_e; e) \left[1/\delta \bar{H}^i(\underline{k}) \hat{Y}(z_e/\delta; i) - \bar{H}^e(\underline{k}) \hat{Y}(z_e; e) \right]}{1 + \kappa^2 \left[\hat{Y}(z_e/\delta; i) + \hat{Y}'(z_e; e) \right]} . \quad 5.5.26$$

Using the leading terms of the expansion of $Y(z;)$ and $Y'(z;)$ for large z and letting δ go to zero, we obtain:

$$\hat{n}_1^e(\underline{k}, z) = \frac{\bar{H}^e(\underline{k})}{\kappa u_e} \hat{Y}(z_e; e) + \frac{1}{\kappa u_e} \frac{\kappa^2 \hat{Y}'(z_e; e) \left[\bar{H}^i(\underline{k})/iz_e - \bar{H}^e(\underline{k}) \hat{Y}(z_e; e) \right]}{1 + \kappa^2 \hat{Y}'(z_e; e)} , \quad 5.5.27$$

or

$$\hat{n}_1^e(\underline{k}, z) = \hat{I}^e(\underline{k}, z) + \frac{\kappa^2 \hat{S}^e(\underline{k}, z) \left[\bar{H}^i(\underline{k})/iz_e - \hat{I}^e(\underline{k}, z) \right]}{1 + \kappa^2 \hat{S}^e(\underline{k}, z)} . \quad 5.5.28$$

This has the well-known dominant conjugate poles at a frequency close to the plasma frequency (provided κ is large and Ω_e/ω_e small). Using the asymptotic expansion of $Y'(z;)$ for large z and assuming that $v_e/\omega_e, \Omega_e/\omega_e$ and $\kappa u_e/\omega_e = 1/\kappa$ are much smaller than one, we find an approximate expression for the location of the poles $z_p = \omega_p + i\gamma$ in the z -plane

$$z_p = \omega_p + i\gamma = \pm \omega_e \left(1 + \frac{1}{2} \frac{(3\kappa^2 u_e^2 + \Omega^2 \cos^2 \alpha)}{\omega_e^2} \right) + i \frac{\gamma}{2} . \quad 5.5.29$$

In the vicinity of this pole, we can approximate the expression for $\hat{n}_e(\underline{k}, z)$ as:

$$\hat{n}_1^e(\underline{k}, z) \approx \frac{\omega_p^2 [\bar{H}^e(k) - \bar{H}^i(k)]}{(z + i\gamma)^2 - \omega_p^2} \frac{1}{iz}, \quad 5.5.30$$

which corresponds to a damped sinusoidal oscillation in the time domain of amplitude $[\bar{H}^e(k) - \bar{H}^i(k)]$. Note that, within the validity of the asymptotic expansion for $\hat{Y}'(z; e)$, collisional damping will override Landau damping terms.

In the case of an arbitrary initial value problem, the excitation of this mode would depend on the arbitrary difference between the ion and electron initial wave amplitude $\bar{H}^i(k)$ and $\bar{H}^e(k)$. But in the fluctuation problem this is fixed, given by (from 5.1.12 and 5.1.13):

$$\bar{H}^e(k) - \bar{H}^i(k) = 1 - \frac{h^2}{h_e^2} \frac{1}{1 + h^2 k^2} - \frac{h^2}{h_i^2} \frac{1}{1 + h^2 k^2} = \frac{h^2 k^2}{1 + h^2 k^2} = \frac{1}{1 + 2\kappa^2}, \quad 5.5.31$$

which is a very small amplitude for the usual case $\kappa^2 \gg 1$ (i.e., large wavelengths as compared to the Debye length) as compared to the low frequency part as given by 5.5.25 (in the $\delta = 0$ approximation) which at $t=0$ has amplitude:

$$\hat{n}_1^e(\underline{k}, t) \Big|_{t=0} = \frac{\kappa^2}{1 + \kappa^2} \bar{H}^i(k) = \frac{1}{2} \left(\frac{\kappa^2}{1 + \kappa^2} \right)^2 \quad 5.5.32$$

or to the total amplitude (at $t=0$ and for any δ) of the wave

$$\hat{n}_1^e(\underline{k}, t) \Big|_{t=0} = 1 - \frac{1}{2} \frac{\kappa^2}{1 + \kappa^2}, \quad 5.5.33$$

which goes to $1/2$ as $\kappa \rightarrow \infty$.

One can combine the two expressions for 5.5.23 and 5.5.26 valid for low and high frequencies and obtain an expression for $\hat{n}_1^e(\underline{k}, z)$ valid for all frequencies,

provided we subtract the singularity at $z=0$ from the high frequency expression which also corresponds to the first term in asymptotic expansion (for large frequencies) of the low frequency term (otherwise the region of intermediate frequencies, $ku_i < z < ku_e$, would be twice accounted for). This expression can be considered as the zeroth order term in an expansion in δ uniformly valid for all frequencies. It is given by:

$$\hat{n}_1^e(\underline{k}, z) = \hat{I}^e(\underline{k}, z) + \frac{\kappa^2 \hat{S}^e(\underline{k}, z) [\bar{H}^i(k) 1/iz - \hat{I}^e(\underline{k}, z)]}{1 + \kappa^2 \hat{S}^e(\underline{k}, z)} - \frac{\kappa^2}{1 + \kappa^2} \frac{\bar{H}^i(k)}{iz} + \frac{\kappa^2}{1 + \kappa^2} \frac{\hat{I}^i(\underline{k}, z)}{1 + \kappa^2 / (1 + \kappa^2) \hat{S}^i(\underline{k}, z)} . \quad 5.5.34$$

Although δ is small, in order for the expansion to be useful, one has to check the order of magnitude of at least the next term in the expansion. For the non-magnetic field case the next term is indeed of order δ . With the magnetic field, one finds that there is some value of α at which the first order terms in the low frequency expansion start to become large. Take for instance, the term $\hat{Y}'(\delta z_i; e) \approx 1 - i\delta z_i m_0 - \delta^2 z_i^2 m_1 + \dots$ and the case $\Omega_e / ku_e \gg 1$, $v/ku_e \ll 1$ which corresponds to typical parameters for ionospheric applications. The function $Y(t; e)$ behaves roughly as $e^{-(t^2 \sin^2 \alpha / 2)}$ with an n^{th} order moment m_n of the order of $(1/\sin \alpha)^{n+1}$. Therefore, the small parameter δ is associated with the factor $1/\sin \alpha$ and the expansion can be considered asymptotic only when $\delta/\sin \alpha \ll 1$. This imposes a condition $\alpha \gg (m_e/m_i)^{1/2}$. One should, therefore, expect the conclusion deduced here from the $\delta = 0$ approximation, mainly the close relationship between the dyna-

mics of the electron wave and the dynamics of free streaming ion waves, to hold only outside this angular range. We shall see from our numerical calculations that this transition does occur at around $\alpha = (m_e/m_i)^{1/2}$. For oxygen this angle is equal to 0.34 degrees. The conclusions about the high frequency behaviour still hold true.

CHAPTER VI

COMPUTING TECHNIQUE AND NUMERICAL RESULTS

1. Computing Technique — A Linear Simulator of Plasma Dynamics.

We shall describe here the technique used to evaluate $\bar{n}_1^e(\underline{k}, t)$ the Laplace inverse of $\hat{n}_1^e(\underline{k}, z)$ as given in 5.1.21. It is basically a simulation technique. The function $\bar{n}_1^e(\underline{k}, t)$ is obtained by simulating in real time a system which has the same dynamics as the electron density we are considering. It avoids having to perform numerically Laplace transforms and their inversion, taking advantage of the fact that the functions $\hat{I}(\underline{k}, z)$ and $\hat{S}(\underline{k}, z)$ are defined in integral representations which are the Laplace transforms of the cumbersome but elementary functions $\bar{I}(\underline{k}, t)$ and $\bar{S}(\underline{k}, t)$. The simulation is performed digitally but we will not be concerned here with the digitalization problems and techniques. Inclusion of such details would take us too far afield.

For the benefit of the reader not familiar with the theory of linear systems we introduce here some of the concepts and properties that we need. The most general relationship between the input $e_1(t)$ and the output $e_o(t)$ in a physically realizable, linear, time invariant system is given by

$$e_o(t) = \int_{-\infty}^{+\infty} e_1(t') h_+(t-t') dt' , \quad 6.1.1$$

where $h_+(t)$ is a characteristic function of the system. The condition of physical realizability imposes a condition on $h_+(t)$: $h_+(t) = 0$ for $t < 0$. This function is referred to as the unit impulse ^{response} of the system since if $e_1(t) = \delta(t)$ then $e_o(t) = h_+(t)$.

Taking the complex Fourier transform of 6.1.1 with a transforming operator $\int_{-\infty}^{\infty} dt e^{-izt}$, with $\Im z$ within a strip of analyticity, we obtain:

$$\hat{e}_o(z) = \hat{h}_+(z) e_1(z) , \quad 6.1.2$$

where $\hat{e}_1(z)$ and $\hat{e}_o(z)$ are the Fourier transforms of the input and output respectively, and $\hat{h}_+(z)$ the Fourier transform of $h_+(t)$. But since $h_+(t) = 0$ for $t < 0$, we can write

$$\hat{h}_+(z) = \int_0^{\infty} dt e^{-izt} h_+(t) , \quad 6.1.3$$

which corresponds to our definition for the Laplace transform used before; we shall, therefore, also refer to it as such. The Laplace transform $\hat{h}_+(z)$ is called the transfer function of the system, and when evaluated for real $z = \omega$ (when it exists) the frequency response of the system. Either the unit impulse response or the transfer function $\hat{h}_+(z)$ completely characterizes a system.

It is standard practice to represent a system schematically by a rectangular box with the unit impulse and or the transfer function inscribed in the center and two arrowed lines representing the input and output.

If several systems are connected in series so that the output of a system is the input of another one can show that the overall transfer function for the new sys-

tem is given by the product of the transfer functions of each of the system elements. If they are connected in parallel, that is, sharing the same input and adding the outputs, the new transfer function is given by the sum of the transfer function of each of the system elements.

If one takes the output of a system and subtracts it from the input, one obtains a new system (an elementary feed-back loop) with a different transfer function. It can easily be shown that the "closed loop" transfer function $\hat{g}_+(z)$ is related to the transfer function of the original system $\hat{h}_+(z)$ (open loop transfer function) by:

$$\hat{g}_+(z) = \frac{\hat{h}_+(z)}{1 + \hat{h}_+(z)} , \quad 6.1.4$$

so that

$$e_o(z) = \frac{\hat{h}_+(z)}{1 + \hat{h}_+(z)} e_1(z) . \quad 6.1.5$$

One can also show that the "error" $e_e(z) = e_i(z) - e_o(z)$ is related to the input by

$$\hat{e}_e(z) = \frac{1}{1 + \hat{h}_+(z)} \hat{e}_1(z) . \quad 6.1.6$$

With above simple rules in mind and by simple inspection of the system represented in Figure 6 one can show that the overall transfer function of such a system $C_+(z)$ is given by:

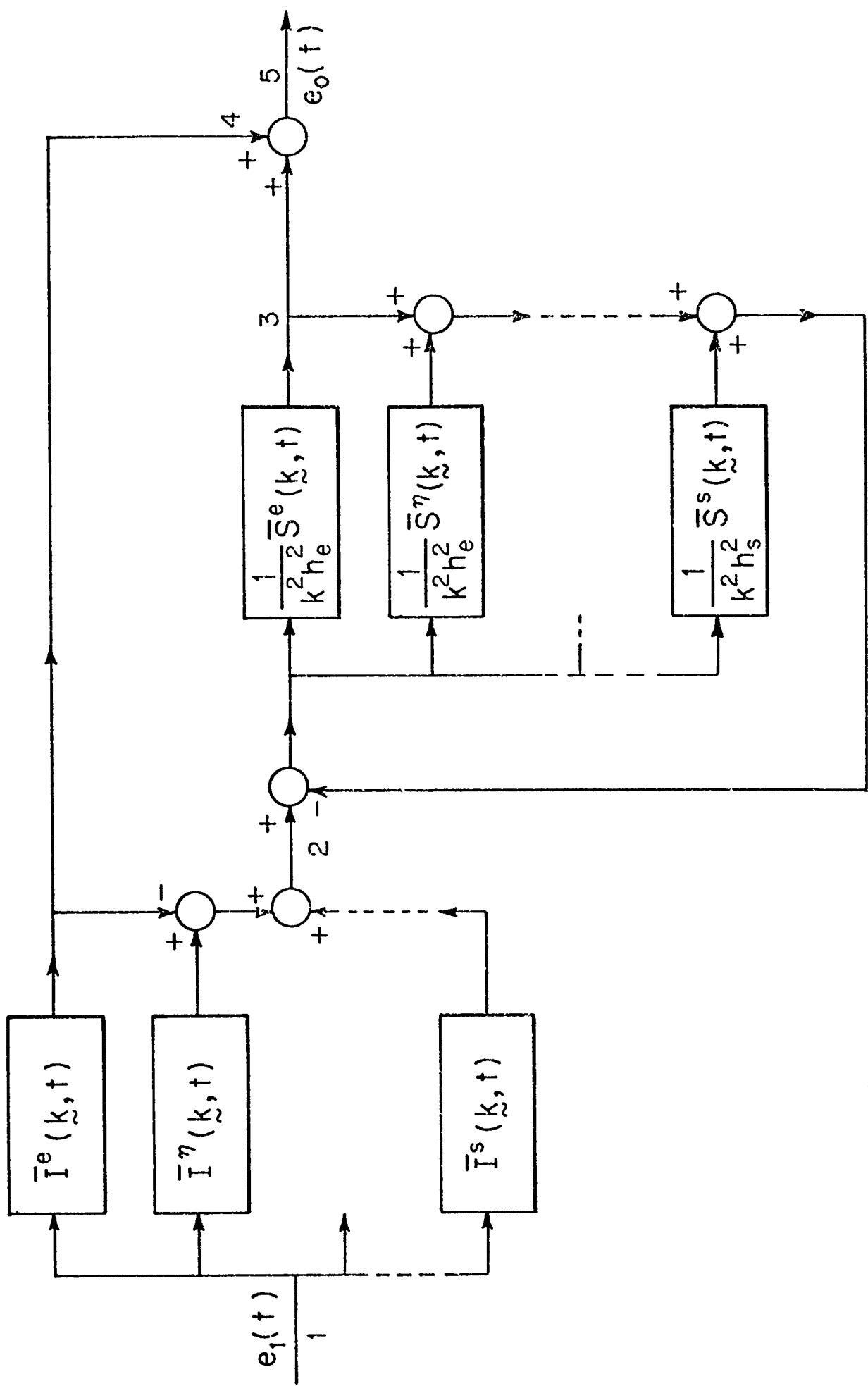


FIG. 6. Schematic diagram of the plasma simulator. It has the same time behaviour as $n_1^e(k, t)$ when excited by a delta function.

$$C_+(z) = \hat{I}^e(\underline{k}, z) + \frac{\frac{1}{k^2 h_e^2} \hat{S}^e(\underline{k}, z) \sum_{\eta}^s Z_{\eta} \hat{I}^e(\underline{k}, z)}{1 + \sum_{\eta}^s \frac{1}{k^2 h_{\eta}^2} \hat{S}^{\eta}(\underline{k}, z)}, \quad 6.1.7$$

which is the expression for $\hat{n}_1^e(\underline{k}, z)$ as given by 5.1.21 ($Z_{\eta} = 1$ for $\eta \neq e$, and $Z_e = -0$). Therefore $\hat{C}_+(z) \equiv \hat{n}_1^e(\underline{k}, z)$ and $\hat{C}_+(t) \equiv \bar{n}_1^e(\underline{k}, t)$.

If we "build" such a system (simulating it numerically or otherwise) and excite it by applying a unit impulse to the input; its time response would give us the desired function $\bar{n}_1^e(\underline{k}, t)$. We are not limited to a unit impulse excitation, this gives up some flexibility which we have taken advantage of and would like to discuss.

In the previous chapter we have seen that, for the wavelengths which we are mainly interested in (i.e., those much larger than the Debye length), most of the features of $\bar{n}_1^e(\underline{k}, t)$ involve time constants of the order of $(ku_i)^{-1}$ or larger, which correspond to the dynamics of the ions. We would like to concentrate on them and filter out higher frequency detail, mainly the plasma frequency oscillations and the electron gyro-frequency. Furthermore, experimentally, for reasons of sensitivity of the instruments, one would like to limit the bandwidth of the system to include only the frequencies of interest. In Chapter II we obtained a relationship between the received signal autocorrelation $R(\tau) = \langle E(t) E(t+\tau) \rangle$ and the density autocorrelation of the medium. There, we implicitly assumed that the system had no bandwidth limitations. If the signal $E(t)$ is fed to a relatively narrow filter characterized by an impulse response $F_{\omega_0}(t)$, then the autocorrelation function

$R'(\tau) = \langle E'(t) E'(t+\tau) \rangle$ of the output $E'(t)$ is given by (see Y.W. Lee³⁰, Statistical Theory of Communications, p. 323)

$$R'(\tau) = F_{\omega_0}(t) * F_{\omega_0}(-t) * R(\tau) , \quad 6.1.8$$

where the asterisks imply a convolution operation. (Notice a double convolution instead of the single one for the signal proper).

Let $C(\tau)$ be an even real function of τ so that

$$\begin{aligned} C(\tau) = C_+(\tau) &= \bar{n}_1^e(k, \tau) & \text{for } \tau > 0 \\ C(\tau) = C_+(-\tau) &= & \text{for } \tau < 0 . \end{aligned} \quad 6.1.9$$

Then we can write $R(\tau)$ as given by 5.1.22 in the form

$$R(\tau) = B C(\tau) \cos \omega_0 \tau \quad \text{for all } \tau \text{'s} , \quad 6.1.10$$

where $B = (|A^2|/2) n_0^e V$. Here we have made use of the evenness of the function $R(\tau)$. For a narrow filter centered around ω_0 the response $F_{\omega_0}(t)$ can be written in the form $F_{\omega_0}(t) = F_+(t) \cos \omega_0 t$, where $F_+(t)$ is a slowly (the slower the narrower the filter is) varying function of time (time scale of the order ω_0^{-1}). Because of the slowness of $F_+(t)$ with respect to $\cos \omega_0 t$ it can be shown that, to a very good approximation:

$$R'(\tau) = B \left[G(\tau) * C(\tau) \right] \cos \omega_0 \tau , \quad 6.1.11$$

where $G(\tau) = F_+(\tau) * F(-\tau)$; an even function of τ .

The function $G(\tau)$ has a certain time width depending on the function $F_+(\tau)$ chosen for the filter. Its effect on $C(\tau)$ is to smear out any high frequency detail in $C(\tau)$ that has a time scale much smaller than the width of $G(\tau)$ and on the other hand to leave all slower frequency detail unmodified (providing we normalize $G(\tau)$, i.e., $\int_{-\infty}^{\infty} dt G(\tau) = 1$).

We can express the new "envelope" of the autocorrelation function $\bar{C}(\tau) = G(\tau) * C(\tau)$ in terms of $C_+(\tau)$. From the definition of $C(\tau)$ and using the fact that $C_+(\tau) = 0$ for $\tau < 0$ we have that:

$$C(\tau) = 2 \text{ Even } C_+(\tau) , \quad 6.1.12$$

where $\text{Even } C_+(\tau)$ implies the operation $\text{Even } C_+(\tau) = 1/2 [C_+(\tau) + C_+(-\tau)]$. And, since $G(\tau)$ is even, we have that

$$\bar{C}(\tau) = 2 \text{ Even } [G(\tau) * C_+(\tau)] . \quad 6.1.13$$

We can obtain $G(\tau) * C_+(\tau)$ from the simulator by applying to it an input* $e_1(t) = G(t)$. The output would then be:

$$e_o(t) = G(\tau) * C_+(\tau) . \quad 6.1.14$$

Besides being able to simulate any filtering that may be present in an experimental set up, the function $G(\tau)$ gives us a controlled maximum bandwidth for the

*

Notice that the inclusion of a filter in series with the (simulator) system will not do since $G(\tau)$ has to be even and therefore, is not physically realizable ($G(\tau) \neq 0$ for $\tau < 0$).

autocorrelation function envelope. In the frequency domain we can write 6.1.14 as

$$\hat{e}_0(\omega) = \hat{G}(\omega) \hat{C}_+(\omega) , \quad 6.1.15$$

where $\hat{e}_0(\omega)$, $\hat{G}(\omega)$ and $\hat{C}_+(\omega)$ are the Fourier transforms (for real $z = \omega$) of $e_0(t)$, $G(t)$ and $C_+(t)$ respectively. This allows us to make some convenient approximations on some of the functions, mainly the electron $\bar{I}^e(\underline{k}, t)$ and $\bar{S}^e(\underline{k}, t)$, before simulation; by smearing out any high frequency detail with frequency higher than the "cut-off" frequency of $\hat{G}(\omega)$, for instance, the electron gyro-frequency. It also determines a minimum and optimum sampling rate in the process of digitalization, since for accurate representation of a function the sampling rate should be at least twice and for practical reasons not much higher than, the "maximum" frequency of the sampled function. It also allows to modify the functions $\bar{S}^e(\underline{k}, t)$ and $\bar{S}^i(\underline{k}, t)$ (mainly the first one) at frequencies higher than the cut-off without introducing appreciable errors in $e_0(t)$. This we have found necessary to overcome some problems of stability.*

We have found the function

$$G(\tau) = \frac{1}{\sqrt{\pi} t_w} \left[\frac{3}{2} - \left(\frac{\tau}{t_w} \right)^2 \right] e^{-(\tau/t_w)^2} , \quad 6.1.16$$

with real Fourier transform:

$$\hat{G}(\omega) = \left[1 + \left(\frac{\omega t_w}{2} \right)^2 \right] e^{-(\omega t_w/2)^2} , \quad 6.1.17$$

* Although the actual system is stable, necessary approximations in the process of digitalization can make the system unstable.

particularly convenient. This function $\hat{G}(\omega)$ is very flat at low frequencies $\omega < t_w^{-1}$ behaving as

$$\hat{G}(\omega) = 1 - \frac{3}{2} \left(\frac{\omega t_w}{2} \right)^4 + 0 \left[\left(\frac{\omega t_w}{2} \right)^6 \right] \quad 6.1.18$$

and cuts-off very sharply at frequencies $\omega > 2t_w^{-1}$. We have used this function in evaluating the functions $\bar{C}(\tau)$ to be described next, using t_w 's no larger than $1/20 (ku/\sqrt{2})^{-1}$ of the lightest of the ion species. It leaves all features due to ion dynamics with frequencies of the order of $ku/\sqrt{2}$ or less almost unmodified and yet provides the desired filtering at higher frequencies.

2. Numerical Results.

Considering the number of parameters involved in the expression for $\bar{n}_1^e(\underline{k}, \tau) = C_+(\tau)$, it is apparent that there is a enormous number of possible combinations that one could discuss. We shall limit them by considering typical ionospheric parameters and selecting a wavelength which corresponds to the Jicamarca Observatory; the only experimental installation capable of observing ion gyro-frequency phenomena, in which we are particularly interested. We have investigated numerically the effect of ion-ion Coulomb collisions, the effects of the angle α ($\alpha = \pi/2 - \theta$, where θ is the angle between \underline{k} and \underline{B}) including very small values of α and the effect of several ion components. The effects of the other parameters: temperature, magnetic field and wavelength are relatively obvious (provided, of course, we keep them within some practical and realistic limits).

We have evaluated the function $\bar{C}(\tau)$ for the conditions specified above and the results are displayed in Figures 7 to 11. By normalizing with respect to $C(0)$ rather than $\bar{C}(0)$, we show explicitly how much "power" has been filtered out, and not accounted for. The signal powers $\langle E^2(t) \rangle$ and $\langle E'^2(t) \rangle$, before and after the filter, are given by $R(0)$ and $R'(0)$, and its ratio $R'(0)/R(0)$ by $\bar{C}(0)/C(0)$. The amount of relative power filtered out is given by

$$\frac{[R(0) - R'(0)]}{R(0)} = 1 - \frac{\bar{C}(0)}{C(0)}, \quad 6.2.1$$

which can be obtained from the plots at $\tau = 0$. In most cases considered, this difference is smaller than the resolution of the plots.

An absolute value for $C(0) = \bar{n}_1^e(\underline{k}, 0)$ can be readily obtained from:

$$C(0) = \bar{n}_1^e(\underline{k}, 0) = H^e(k) = 1 - \frac{h^2}{h_e^2} \frac{1}{1 + k^2 h^2}, \quad 6.2.2$$

this in turn can be considered as a measure of the relative power of the scattered signal as compared to the power received in the hypothetical case in which there is no correlation between the particles (Thomson-Rayleigh scattering), in which case $H^e(k) = 1$. For all cases considered here $k^2 h^2 \ll 1$ and $C(0) \approx 1/2$.

Figures 7a, b, and c show the autocorrelation function envelope, $\bar{C}(\tau)$, for hydrogen, helium, and oxygen ions respectively, under similar conditions and for different angles of the wave vector and the magnetic field (α in degrees from perpendicular). Collisions have not been included here ($\nu = 0$) for the purpose of discussing only the effects of the angle α . The normalized time is equal to $ku_1/\sqrt{2}t$.

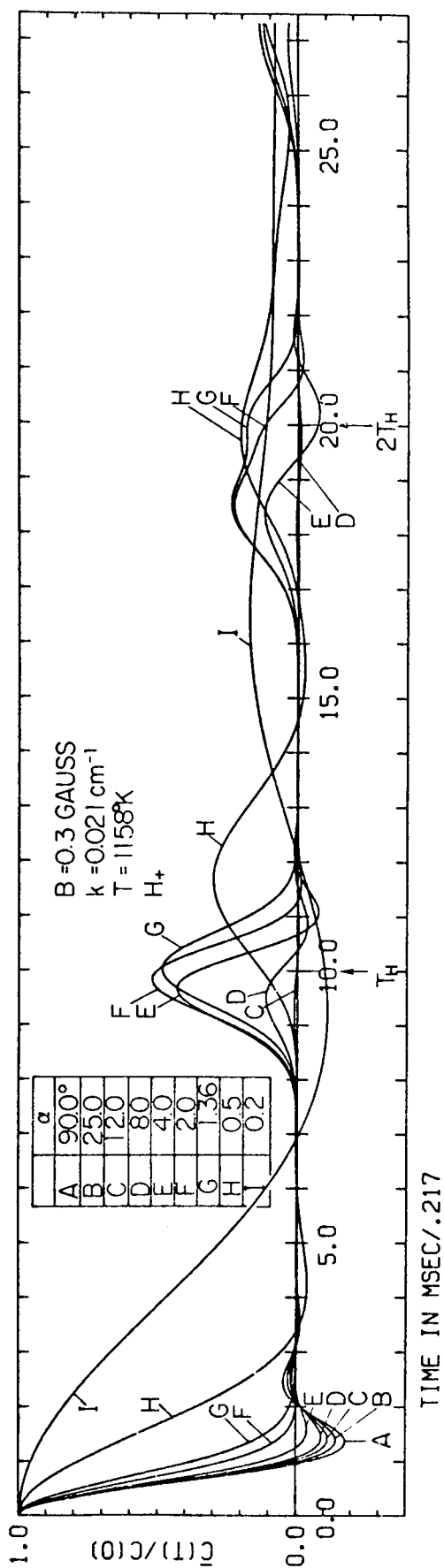


FIG. 7a. Autocorrelation function envelope $\bar{C}(\tau)$ for a $[\text{H}]^+$ plasma showing the effect of variations in the direction of the wave vector \tilde{k} as it approaches the perpendicular to magnetic field ($\alpha = 0$). Collisionless model, other parameters as shown in the figure.

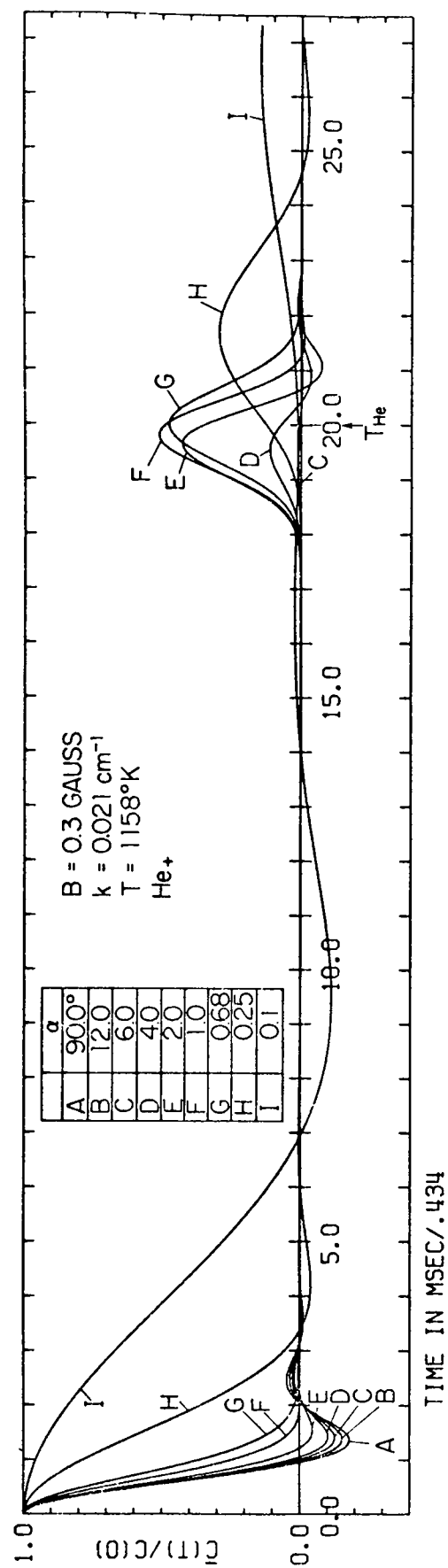


FIG. 7b. Same as Fig. 7a but for $[\text{He}]^+$

T

We have used the values for \underline{B} , \underline{k} , and t shown in the figure and a density of $n^e = 5 \times 10^5$, but, provided one stays in the range of small $k^2 h^2$, the curves are insensitive to density variations and could correspond to any density higher than $\approx 10^3 \text{ cm}^{-3}$. The first point to notice is the qualitative similarity between these functions and $\bar{I}^i(\underline{k}, t)$ (Figure 3) for angles (in radians) larger than $(m_e/m_i)^{1/2}$ (this critical angle occurs at 1.34, 0.68, and 0.34 for H^+ , H_e^+ , and O^+ respectively), the main differences being in the slight oscillatory tendency of $\bar{C}(\tau)$ as compared to $\bar{I}^i(\underline{k}, \tau)$ (see Figure 3) and a shift toward the left of the gyro-periods (labeled as T_H, T_{H_e} , and T_O) for the occurrence of the maximums in the auto-correlation function. Curves for the different species labeled with the same letter correspond to angles which are in the same proportion as the square root of the ratio of their masses and show a very close similarity to each other. Notice that the gyro-peaks show up only for angles smaller than the ones corresponding to curve C. As far as the behaviour for times less than the gyro-frequency there is little change until the angle is reduced to the small value corresponding to curve B.

At exactly the critical angle, $\alpha = \sin^{-1}(m_e/m_i)^{1/2}$ (curve G) a peculiar thing happens, in that the behaviour of $\bar{C}(\tau)$ for small times is almost exactly that of $\bar{I}^i(\underline{k}, \tau)$. The reason is that at this angle ($\Omega^e \gg k u^e$) we have that

$$\bar{I}^e(\underline{k}, t) \approx H^e(k) e^{-k^2 u_e^2 \sin^2 \alpha / 2} = H^e(k) e^{-k^2 u_i^2 t^2 / 2}, \quad 6.2.3$$

and for times close to the origin we have

$$I^i(\underline{k}, t) \approx H^i(\underline{k}) e^{-k^2 u_i^2 t^2 / 2}; \quad 6.2.4$$

therefore, the input to the feed-back loop (point 2 in Figure 6) proportional to their difference is almost zero (since $H^e(\underline{k}) \approx H^i(\underline{k})$) and the only significant contribution to the output is from $\bar{I}^e(\underline{k}, t) \approx \bar{I}^i(\underline{k}, t)$.

For smaller angles than the critical one, the dynamics of the electrons, become important and the roles of ions and electrons (for times less than gyro-period) are reversed. (Notice the similarity between the curves A and I when the time scale of the latter is adjusted by a factor $(m_e/m_i \sin \alpha)^{1/2}$). The electrons behave like heavy elements, because of the constraint to move only along the magnetic field line, with equivalent mass $m_e/\sin \alpha$ in a sea of neutralizing lighter particles (the ions). At these angles the "gyro-peaks" are broadened and shifted to the right, this can also be explained in terms of apparent heavy electrons, since then one would expect the electrons to try to follow the ions but with much higher inertia, and then, to take a longer time to diffuse out.

Figures 8a, b, and c, show the effect of collisions. Here, keeping the angle and other parameters constant, we have varied the density and used a corresponding collision frequency in accordance with the formula obtained in Appendix I. The parameters selected, other than density, are typical for the ionosphere and a wavelength corresponding to back-scatter at 50 Mc as used by the Jicamarca Observatory. The curves show that for the typical ionospheric concentrations of He^+ and H^+ of the order of 10^{+4} cm^{-3} or less, one would expect negligible attenuation of the

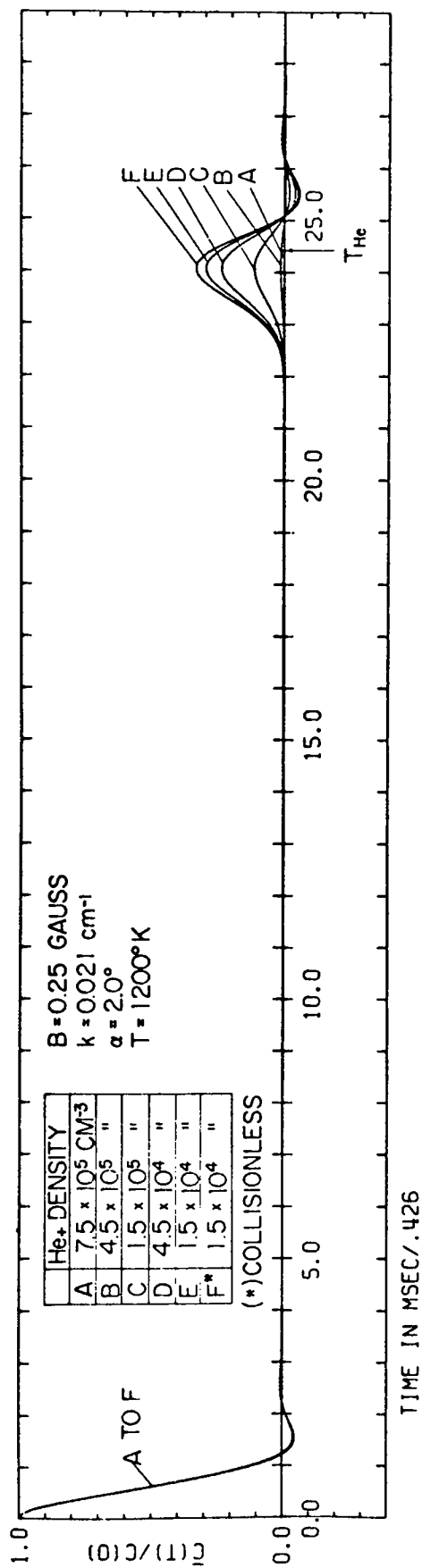


FIG. 8b. Same as Fig. 8a but for $[\text{He}]^+$

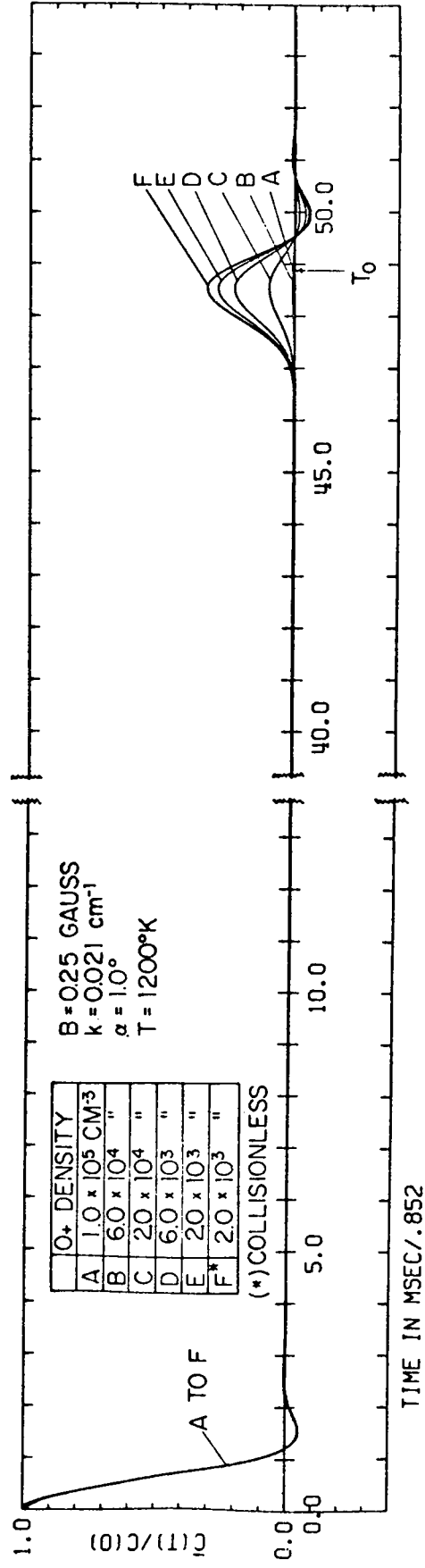


FIG. 8c. Same as Fig. 8a but for $[\text{O}]^+$

"gyro-peaks" corresponding to these constituents. On the other hand, one should not expect to "see" the oxygen peaks for typical concentrations between 10^4 and 10^6 cm^{-3} unless one goes to sufficiently high altitudes where the density drops to values of the order of or less than 10^{+4} cm^{-3} . The amount of attenuation with respect to the collisionless case is very close to the one expected from the behaviour of $\bar{I}^i(k,t)$ and for practical purposes one could use

$$\frac{\bar{C}(t_g; \nu)}{\bar{C}(t_g; 0)} = e^{-4\pi(\nu_i/\Omega)(k^2 u_i^2/\Omega_i^2)} \quad 6.2.5$$

as a semi-empirical formula to estimate the amount of attenuation at the first "gyro-peak."

It is interesting to note, taking a specific example, that for curve B for oxygen (Figure 8c), the characteristic numbers ν_i/ku_i and ν_i/Ω_i are only 2.1×10^{-4} and 2.32×10^{-3} , and yet, the effects of collisions are sufficiently high to attenuate the amplitude of the gyro-peak almost completely.

Figure 9a to f show the effects of different ion composition. They are grouped in sets of equal amounts of oxygen. A density of $3 \times 10^4 \text{ cm}^{-3}$ has been selected and the effect of collisions is included. Other parameters as indicated in the figure. The behaviour is as one would expect from physical grounds. Notice the almost linear relationship between the amplitude of the gyro-peaks and the percentage of the corresponding constituents. The use of incoherent scattering technique as an instrument to measure the ion composition in the ionosphere is evident.

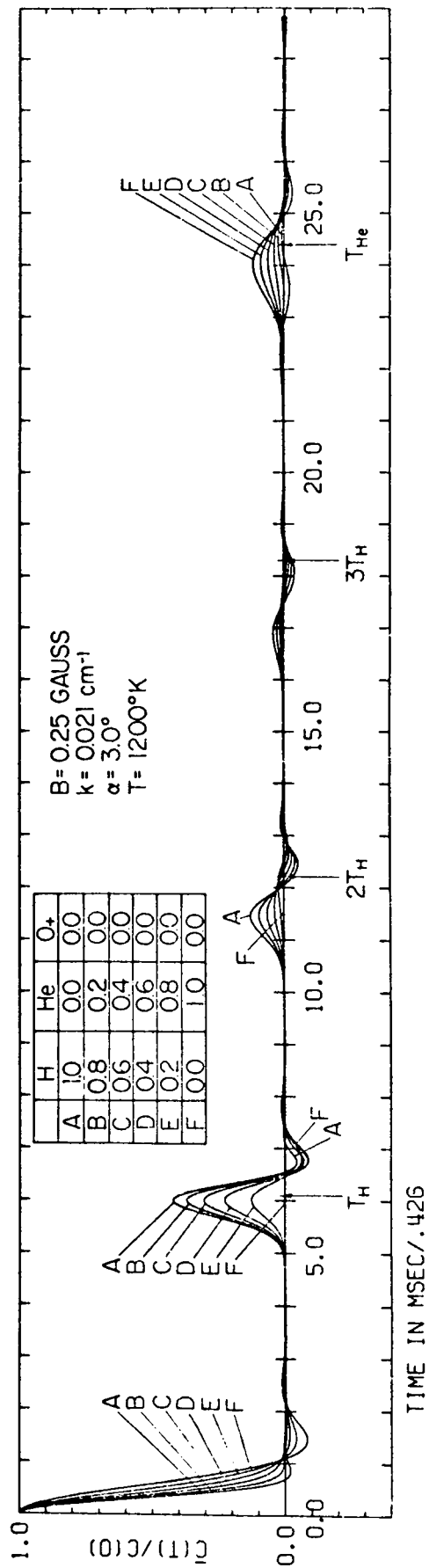


FIG. 9a. Autocorrelation function envelope $\bar{C}(\tau)$ showing the effect of different composition. Here, we keep $[O]^+$ concentration at 0 %. Effects of collisions included, other parameters as shown in figure.

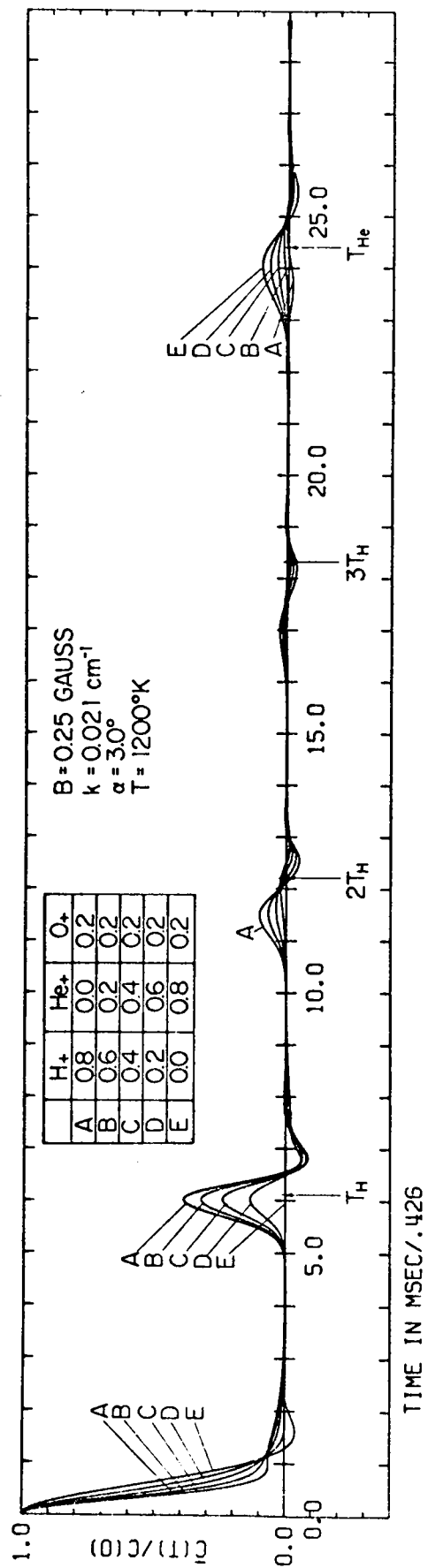


FIG. 9b. Same as Fig. 9a but for $[\text{O}]^+$ concentration of 20%

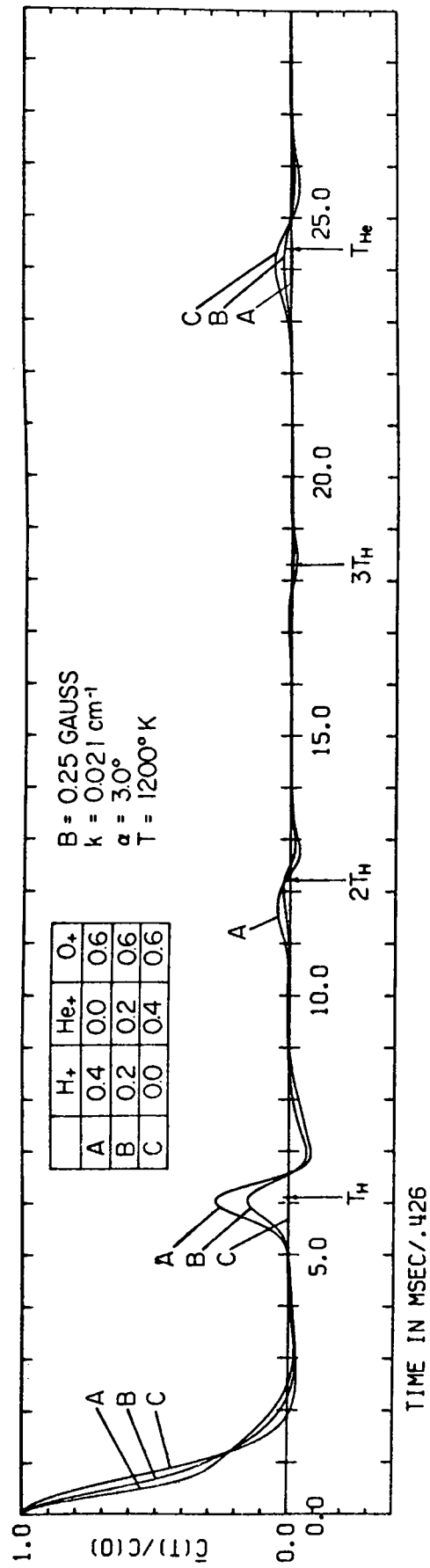


FIG. 9d. Same as Fig. 9a but for $[O]^+$ concentration of 60%

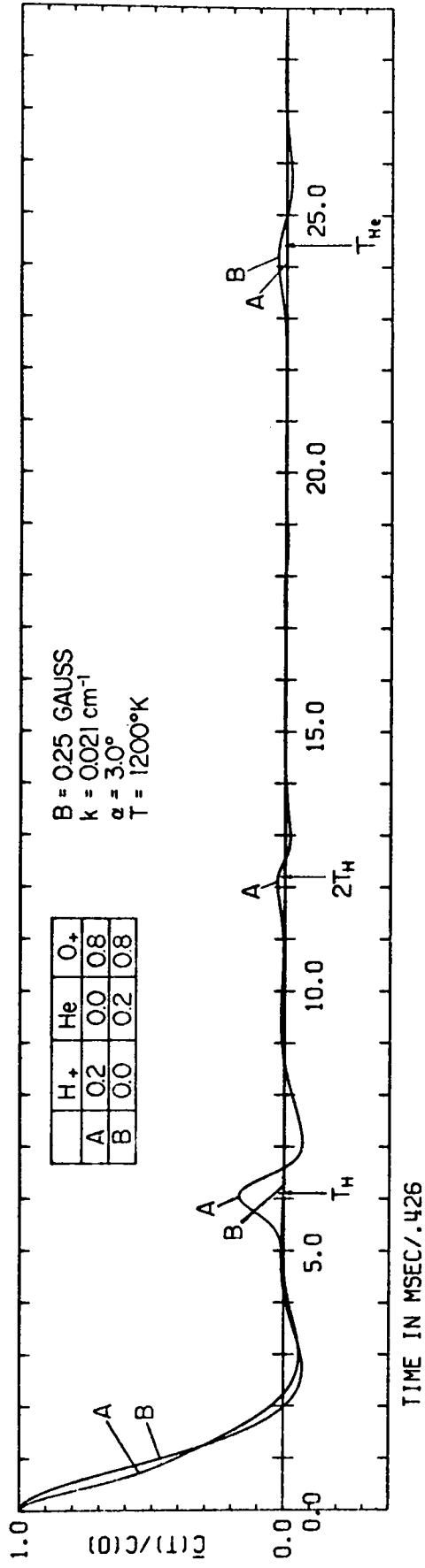


FIG. 9e Same as Fig. 9a but for $[\text{O}]^+$ concentration of 80%

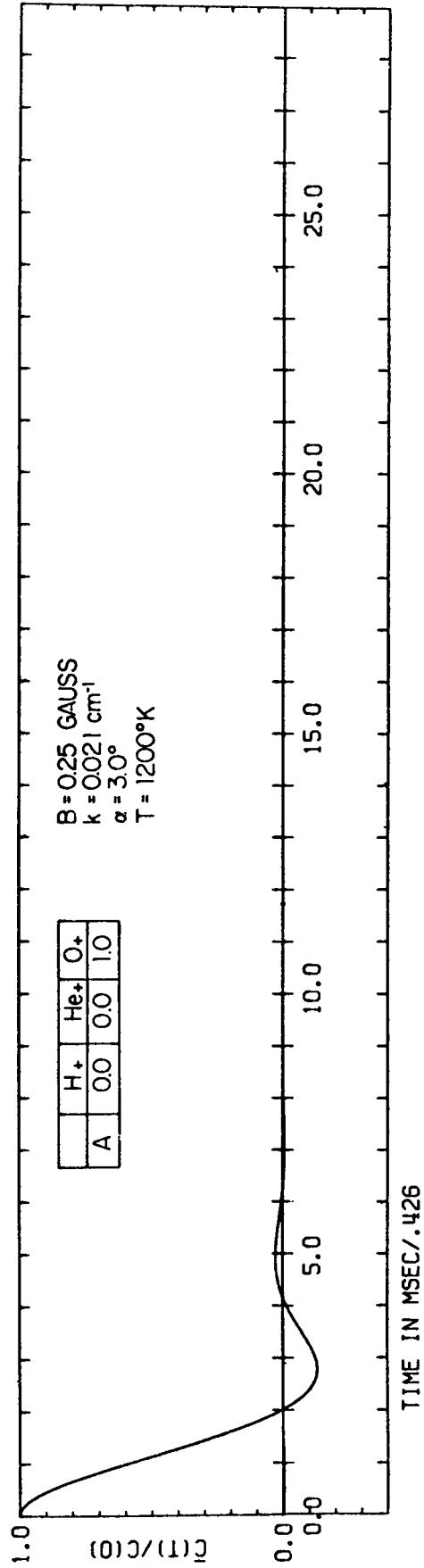


FIG. 9f. Same as Fig. 9a but for $[O]^+$ concentration of 100%

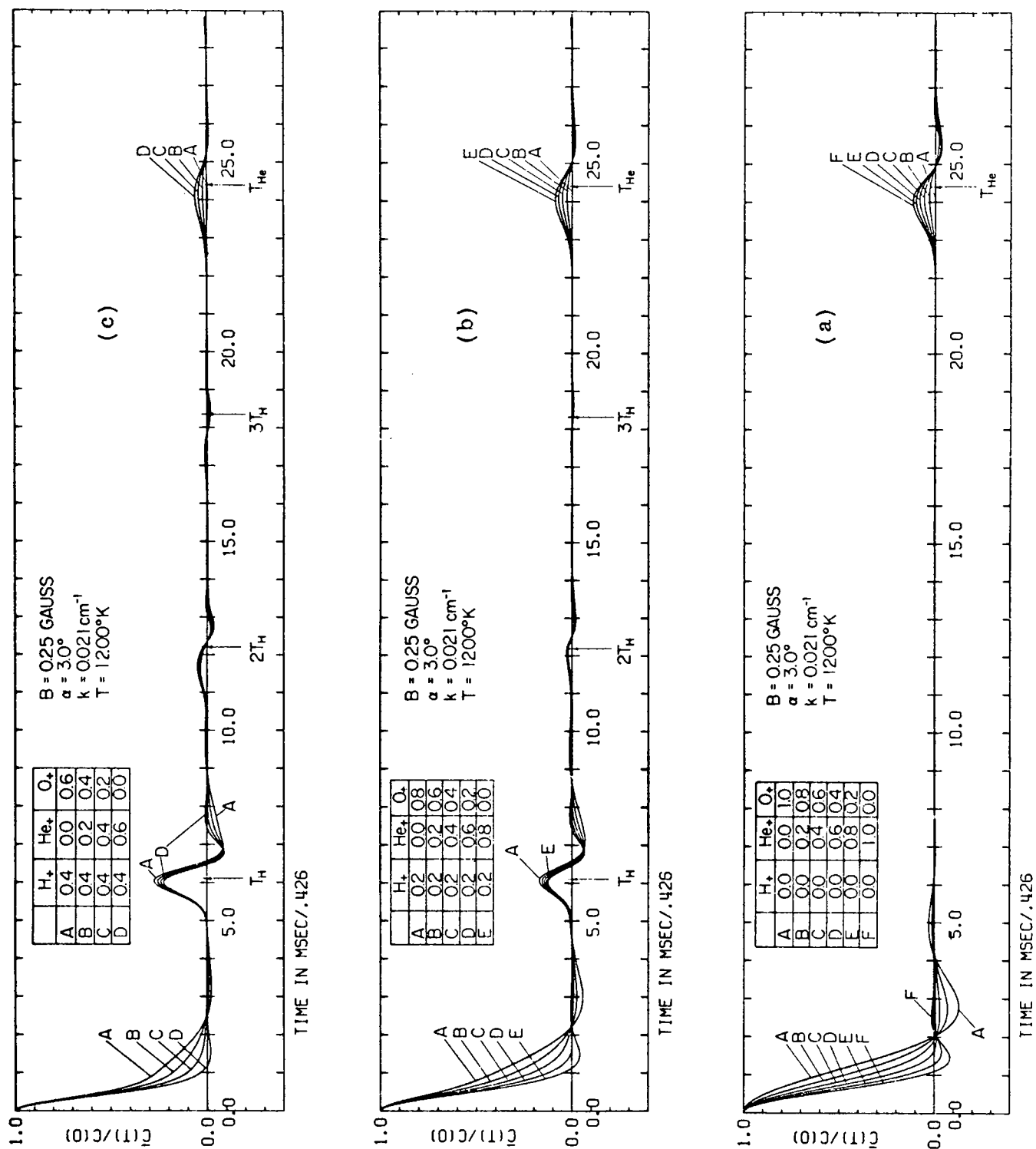


FIG. 10a-c. Same curves as shown in Fig. 9a to f but keeping $[\text{H}]^+$ concentration constant

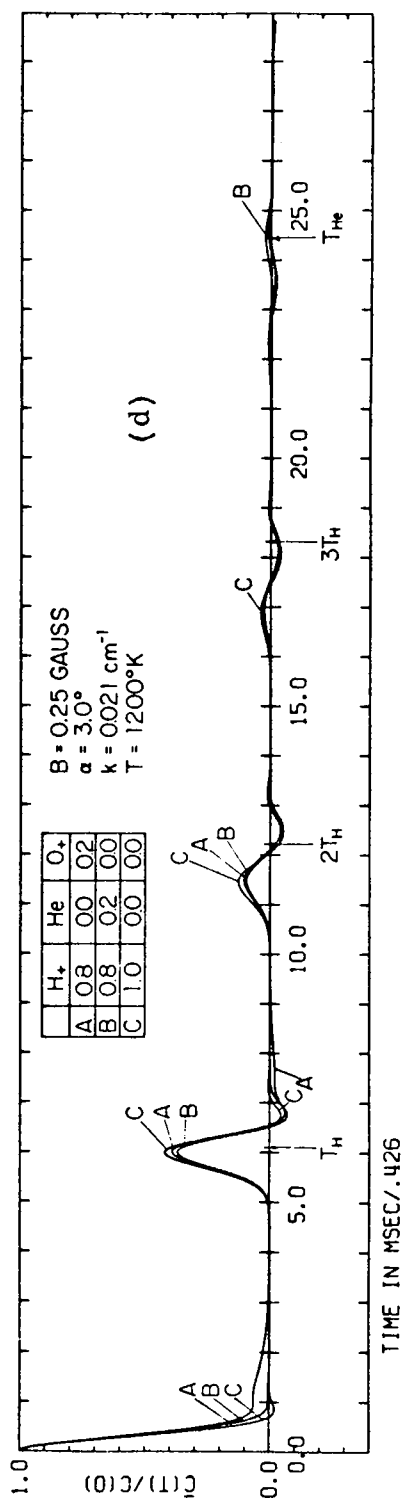
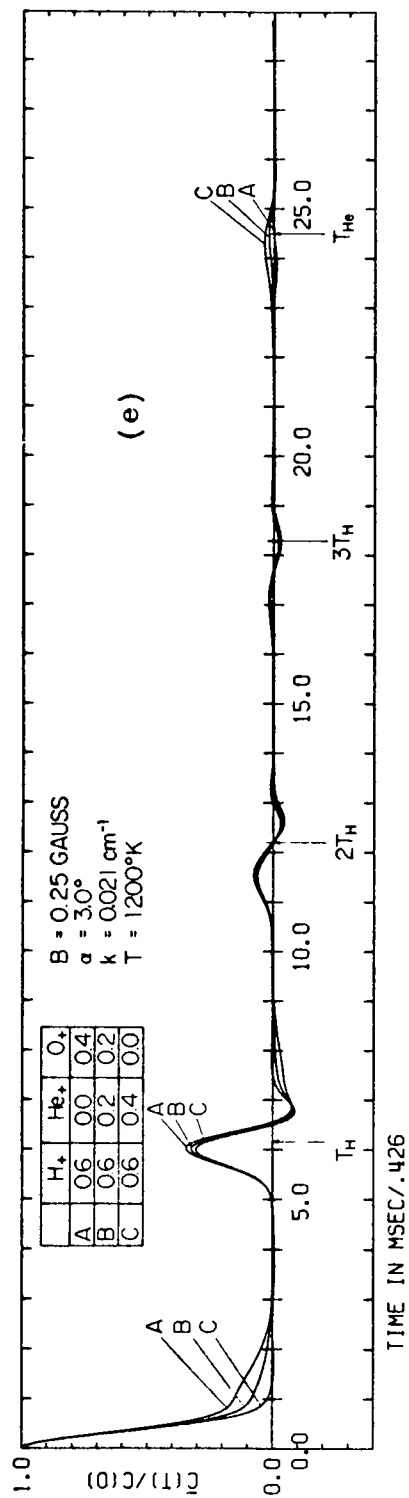


FIG. 10 d and e. Same as Fig. 10a-c

Figures 10a to e, show the same curves as Figure 9 but grouping them for a constant amount of hydrogen. The combination of both sets, Figures 9 and 10, become useful when fitting experimental data. Notice the relative constant amplitude of the hydrogen gyro-peaks as the compositions of the other elements is varied.

3. Experimental Verification.

For that part of the autocorrelation function close to the origin there is a considerable amount of experimental verification with existent collisionless theories, which are sufficient in this time range, as shown by the insensitivity of this part of the curves to the inclusion of collision effects in Figure 8). Figure 11 shows experimental points as obtained in Jicamarca compared with theoretical curves at bounding temperatures (after Farley)³¹. Curves obtained using our theoretical and numerical results agree with the ones given here by Farley. Clearly a temperature of 2000° with a composition of 40% O⁺ and 60% H⁺ would produce a very good fit.

It has not been until recently that the "gyro-peaks" have been observed and even now there are observations only for hydrogen. Although this is not the best element to check the collision model it does provide a good check for the theory in general. Figures 12 and 13 show those recent experimental observations at Jicamarca, Peru (Farley — personal communication). Figure 14 shows the same experimental points shown in Figure 13 for 750 km (note they are taken at different

times and thus not necessarily at the same composition) and the theoretical curve for the expected value of the magnetic field, with composition and temperature adjusted for best fit.

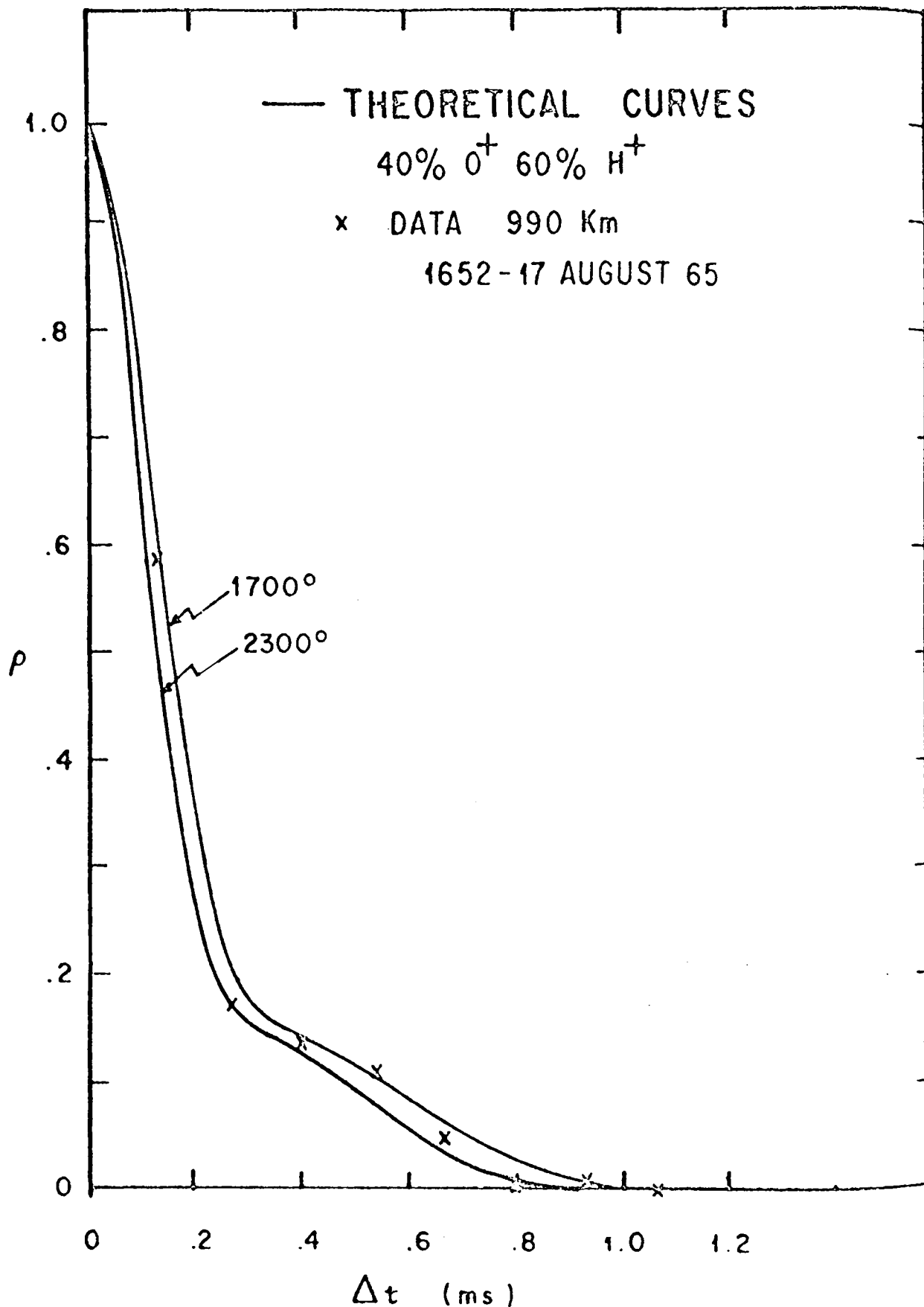


FIG. 11. Experimental points obtained at Jicamarca for time delays less than gyro-period showing agreement with theory (After Farley reference 31).

JICAMARCA PERU

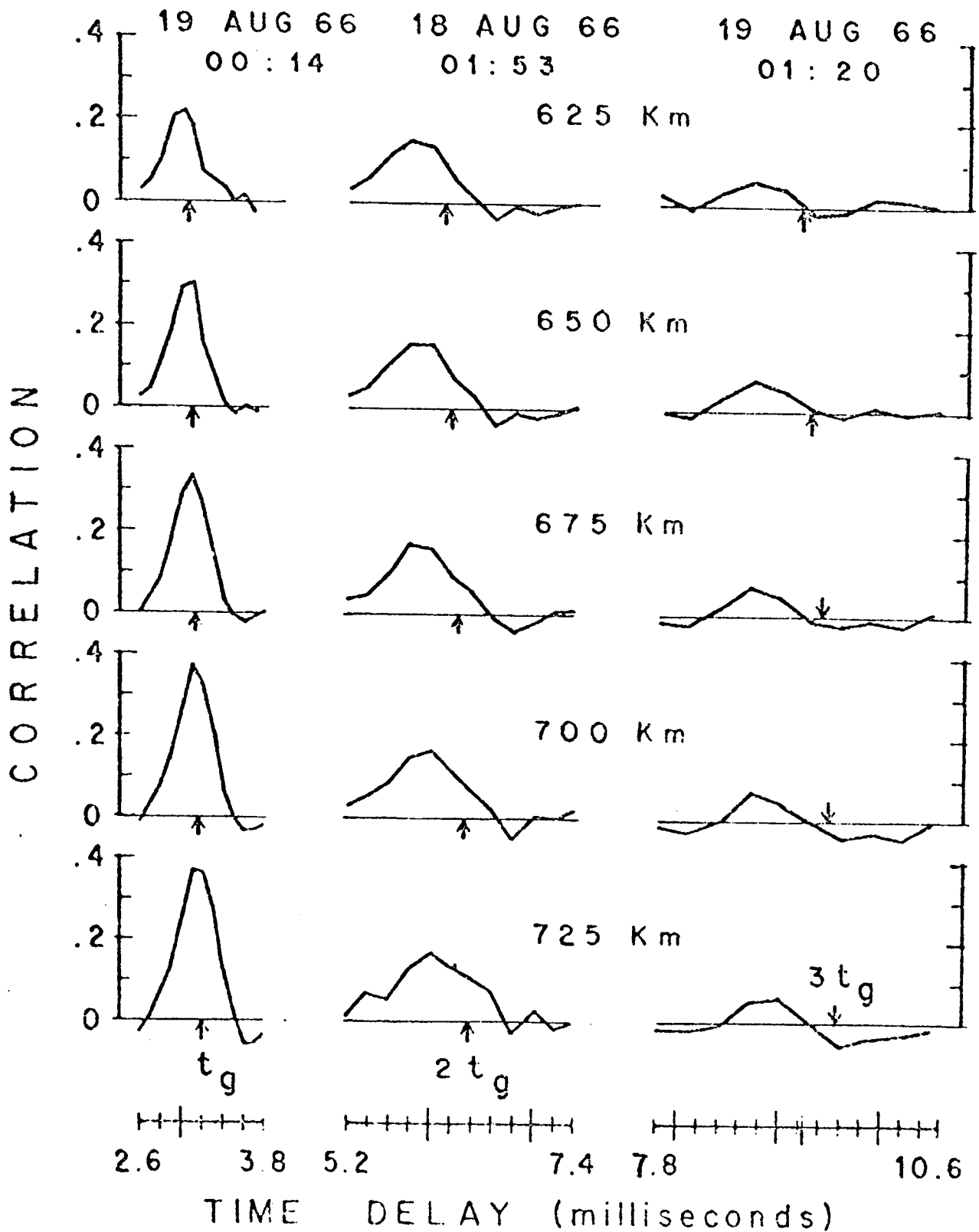


FIG. 12. Experimental curves showing autocorrelation peaks at multiples of the gyro-periods of hydrogen (After Farley personal communication).

JICAMARCA, PERU

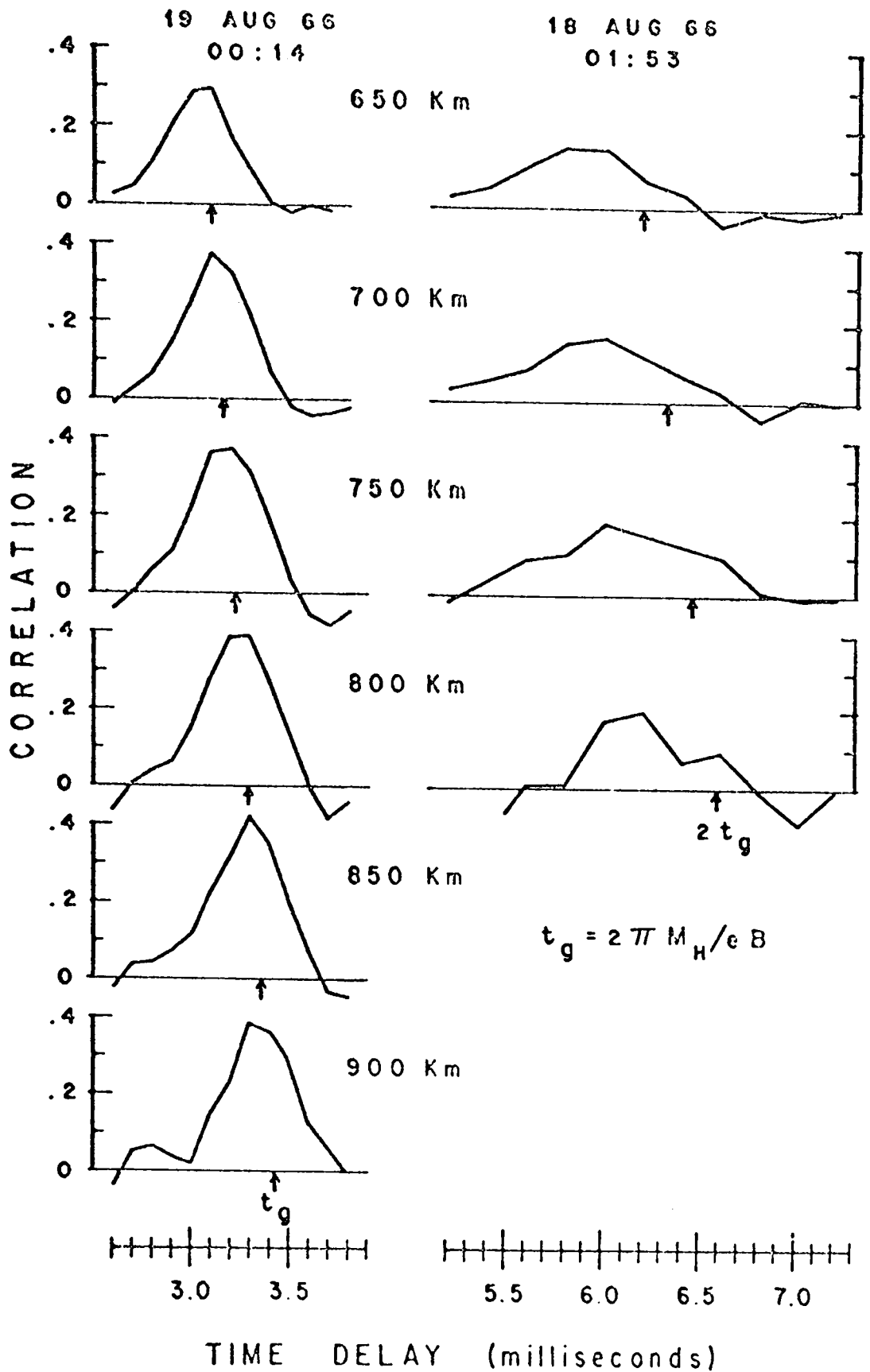


FIG. 13. Same as Fig. 12 but up to higher altitudes

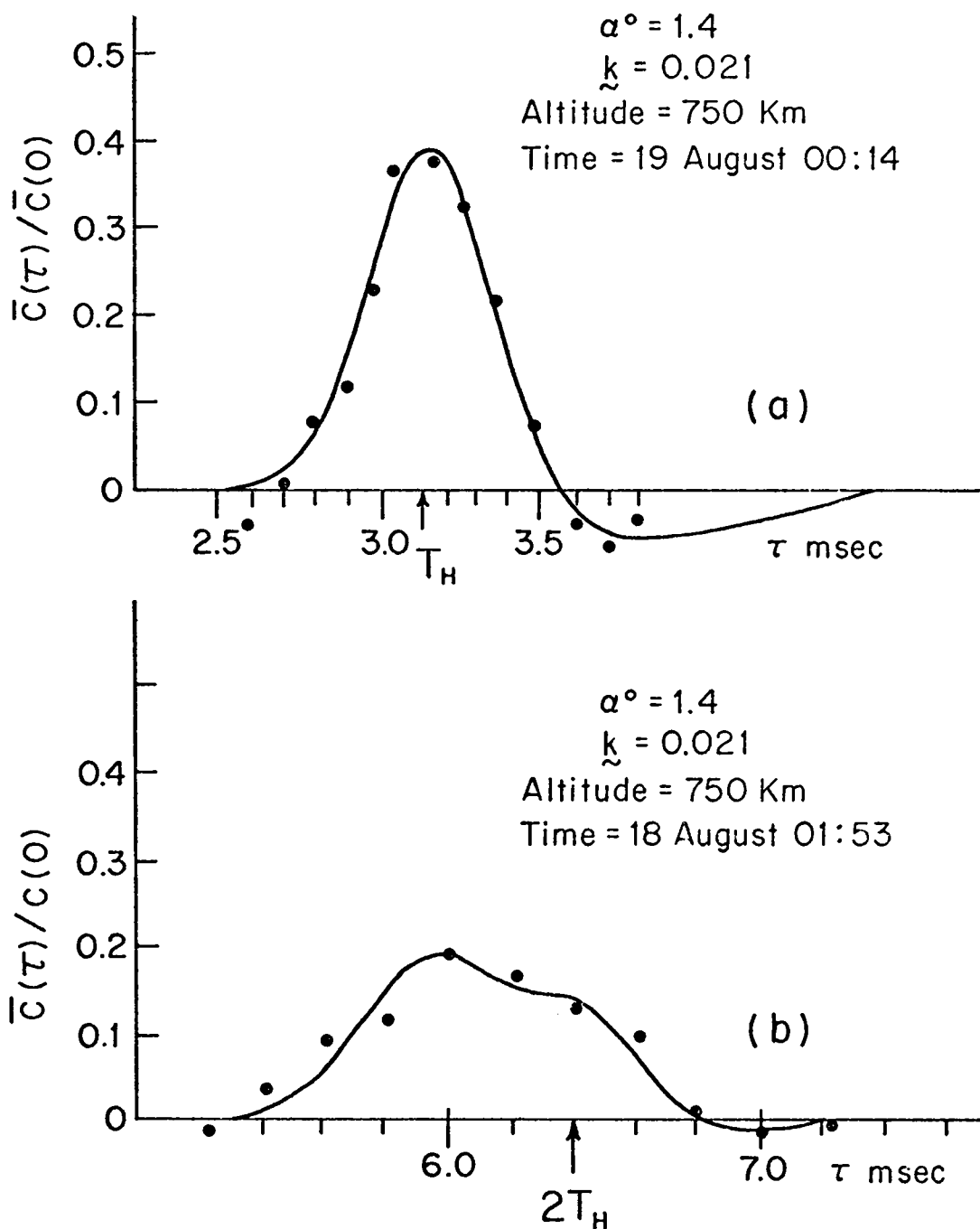


FIG. 14. Experimental points from Fig. 13 and theoretical curves with unknown parameters (temperature and composition) adjusted for best fit. Magnetic field has also been connected slightly ($\sim 2\%$) from theoretical models for best fit. Composition used is 60% $[H]^+$ and 40% $[O]^+$ for Fig. a and 90% $[H]^+$ and 10% $[O]^+$ for Fig. b, both at 700°K.

APPENDIX I

A more accurate collision term than the one we have used in the text is that obtained by M. Rosenbluth, W.M. MacDonald and D.L. Judd.³² We shall use it to derive an appropriate value for the "collision frequencies," ν_μ used in our equations. They obtain a collision term $(\partial f^\mu / \partial t)_{\text{coll}}$ of the Fokker-Planck form, viz.:

$$\frac{1}{\Gamma_\mu} \left(\frac{\partial f^\mu}{\partial t} \right)_{\text{coll}} = - \frac{\partial}{\partial v_i} \left(f^\mu \frac{\partial h^\mu}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f^\mu \frac{\partial^2}{\partial v_i \partial v_j} g^\mu \right), \quad \text{I-1}$$

where

$$h^\mu = \sum_{\eta}^s \frac{m_\mu + m_\eta}{m_\eta} \int d\tilde{v}' f^\eta(\tilde{v}') |\tilde{v} - \tilde{v}'|^{-1}, \quad \text{I-2}$$

$$g^\mu = \sum_{\eta}^s \int d\tilde{v}' f^\eta(\tilde{v}') |\tilde{v} - \tilde{v}'|, \quad \text{I-3}$$

$$\Gamma_\mu = \frac{4\pi e^4}{m_\mu^2} \ln(24\pi n_e h^3), \quad \text{I-4}$$

and $f^\mu \equiv f^\mu(\tilde{v}) \equiv f^\mu(\underline{x}, \underline{v}, t)$ is the local distribution function. Summation over the indexes i and j is implied.

This equation was derived for the case of non-magnetic field. But, if we

assume that the gyro-radius is much larger than the Debye length, it is reasonable to expect that the dynamics of the collisions are not going to differ much, since the trajectories of the particles within the range of interaction (i.e., within a Debye length) would not differ much from the non-magnetic field case. We shall therefore, be limited to the condition $h \ll r_g$ which includes ionospheric plasmas.

Following a perturbation expansion $f^\mu = n_\mu \varphi^\mu + f_1^\mu$ ($\varphi \equiv \varphi(\underline{v})$ stands for a Maxwellian distribution), keeping only the terms involving f_1^μ and neglecting the ones involving any moments of it as we did for our simplified model we obtain:

$$\left(\frac{\partial f_1^\mu}{\partial t} \right) = - \frac{\partial}{\partial v_i} \left(f_1^\mu \Gamma_\mu \frac{\partial h_o^\mu}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f_1^\mu \Gamma_\mu \frac{\partial^2 g_o^\mu}{\partial v_i \partial v_j} \right), \quad \text{I-5}$$

where

$$h_o^\mu = \sum_{\eta}^s n_{\eta} \frac{(m_\mu + m_{\eta})}{m_{\eta}} \int d\underline{v}' \varphi^{\eta}(\underline{v}') |\underline{v} - \underline{v}'|^{-1}, \quad \text{I-6}$$

$$g_o^\mu = \sum_{\eta}^s n_{\eta} \int d\underline{v}' \varphi^{\eta}(\underline{v}') |\underline{v} - \underline{v}'|. \quad \text{I-7}$$

By carrying out the integrations and differentiation we can express the coefficients $\Gamma_\mu (\partial h_o^\mu / \partial v_i)$ and $\Gamma_\mu (\partial^2 g_o^\mu / \partial v_i \partial v_j)$ in terms of two functions $A_1(x)$ and $A_2(x)$ defined by:

$$A_1(x) = \frac{1}{x^3} \left[\text{erf}(x) - \frac{2}{\sqrt{\pi}} x e^{-x^2} \right], \quad \text{I-8}$$

$$A_2(x) = \frac{1}{2x^3} \left[(2x^2 - 1) \text{erf}(x) + \frac{2}{\sqrt{\pi}} x e^{-x^2} \right], \quad \text{I-9}$$

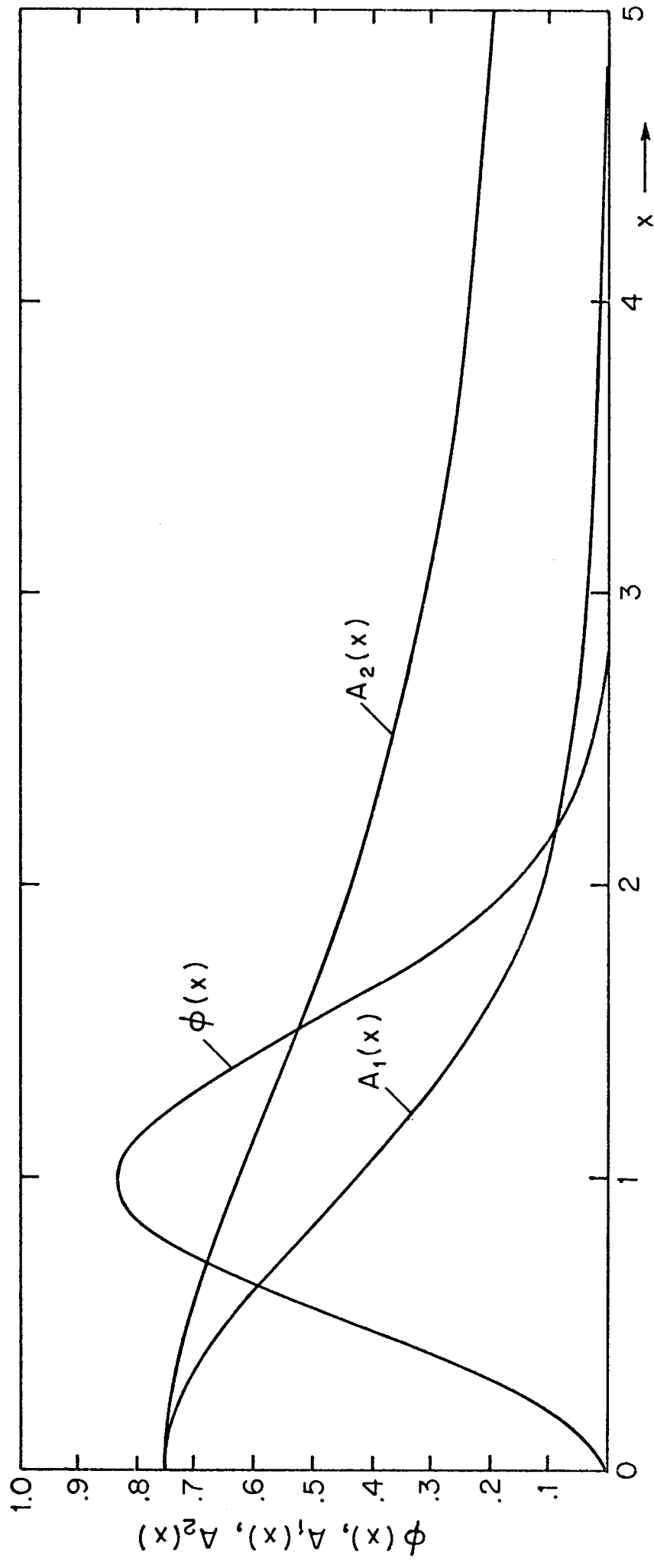


FIG. I-1. The functions $\phi(x)$, $A_1(x)$ and $A_2(x)$.

and plotted in Figure 15, so that:

$$-\Gamma_{\mu} \frac{\partial h_0^{\mu}}{\partial v_i} = \Gamma_{\mu} \sum_{\eta}^s \frac{n_{\eta} (m_{\mu} + m_{\eta})}{m_{\eta}} \frac{1}{u_{\eta}^3} A_1\left(\frac{v}{u_{\eta}}\right) v_i, \quad \text{I-10}$$

$$\Gamma_{\mu} \frac{\partial^2 g_0^{\mu}}{\partial v_i \partial v_j} = \Gamma_{\mu} \sum_{\eta}^s \frac{n_{\eta}}{u_{\eta}} \left\{ A_2\left(\frac{v}{u_{\eta}}\right) \delta_{ij} - \left[A_2\left(\frac{v}{u_{\eta}}\right) - A_1\left(\frac{v}{u_{\eta}}\right) \right] \frac{v_i v_j}{v^2} \right\}. \quad \text{I-11}$$

These terms correspond to $\nu_{\mu} v_i$ and $\nu_{\mu} \frac{u_{\mu}^2}{\mu} \delta_{ij}$ in our simplified model.

They are referred to as the frictional and dispersion coefficient respectively since the first is responsible for the slowing down and the second for the probabilistic dispersion of a test particle.

As far as our particular problem is concerned, we have seen in the main text that the most important effect of collisions is that of reducing the amplitude of the "gyro-peaks." This was explained in terms of the probabilistic dispersion suffered by the particles and for which the dispersion term $\frac{1}{2} \nu_{\mu} \frac{u_{\mu}^2}{\mu} (\partial^2 f^{\mu} / \partial v_i \partial v_j)$ is mainly responsible. We would like to find a value of ν_{μ} so that the dispersion coefficient $\frac{1}{2} \nu_{\mu} \frac{u_{\mu}^2}{\mu}$ be the best approximation to the more accurate coefficient given by I-11. We have to ignore the anisotropic terms in this coefficient because of our isotropic model. It would have been desirable to use an approximate model with an anisotropic part but the complexity of the problem increases considerably. If we are going to replace the velocity dependent coefficients $A_2(v/u_{\eta})$ by a constant, clearly this should be a value of $A_2(v/u_{\eta})$ corresponding to those velocities

for which the particle population of the gas in question, μ , is the largest. We obtain such a characteristic value by averaging $A_2(v/u_\eta)$ over all velocities weighted by the velocity distribution $\varphi^\mu(\underline{v})$. Following this criteria we obtain a value for v_μ given by:

$$v_\mu = \frac{\Gamma_\mu}{u_\mu^3} \sum_{\eta}^s n_\eta C_{\mu\eta} \quad , \tag{I-12}$$

where

$$C_{\mu\eta} = \left(\frac{m_\eta}{m_\mu}\right)^{1/2} \int d^3\underline{v} \varphi^\mu(\underline{v}) A_2\left(\frac{v}{u_\eta}\right) = \left(\frac{m_\eta}{m_\mu}\right)^{1/2} \frac{4}{\sqrt{\pi}} \int_0^\infty x^2 e^{-x^2} A_2\left[\left(\frac{m_\eta}{m_\mu}\right)^{1/2} x\right] dx \quad . \tag{I-13}$$

The value of $C_{\mu\eta}$ is non-dimensional and depends only on the ratio m_η/m_μ . We have evaluated it numerically for the masses of the electron, hydrogen, helium, and oxygen, and the corresponding values are shown in Table I-1.

TABLE I-1
Values of the Constant $C_{\mu\eta}$

$\mu \backslash \eta$	Hydrogen	Helium	Oxygen	Electron
Hydrogen	0.601	0.853	1.015	0.0176
Helium	0.352	0.601	0.854	0.0090
Oxygen	0.185	0.351	0.601	0.0048
Electron	1.127	1.128	1.128	0.601

In c.g.s. units I-12 takes the form:

$$\nu_{\mu} = 0.113 \frac{\ln \Lambda}{M_{\mu}^{1/2} T^{3/2}} \sum_{\eta} n_{\eta} C_{\mu\eta},$$

where M_{μ} is the mass of the μ particles in a.m.u. and $\Lambda = 24\pi n_{\mu} h^3$.

In the case of a single (ion) component plasma it reduces (neglecting the effect of the electrons) to

$$\nu_{\mu} = 0.0680 \frac{n_{\mu} \ln \Lambda}{m^{1/2} T^{3/2}},$$

which corresponds to the inverse of t_c as given by Spitzer.³³ He gives a coefficient equal to 0.0877 instead of 0.0680, this is due to the fact that he takes a value of $v = (1.5)^{1/2} u_{\mu}$ to obtain a characteristic value for $A_1(v/u_{\mu})$ whereas we take an average which weights $v = u_{\mu}$ more strongly.

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